# Characterizing the Best Ranked Set Sampling Scheme via the Fisher Information

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# Contents

# Symbols

| | |
|---|---|
| c.v. | Coefficient of variation. |
| $N(\theta, 1)$ | Normal distribution with known variance. |
| $N(0, \sigma^2)$ | Normal distribution with known mean. |
| $N(\theta,\ \theta^2)$ | Normal Distribution with Constant Coefficient of Variation |
| Logistic$(\theta, 1)$ | Logistic distribution with known scale parameter. |
| Logistic$(1, \sigma)$ | Logistic with known location parameter. |
| $\exp(\theta)$ | Exponential Distribution. |
| Cauchy $(\theta, 1)$ | Cauchy distribution with known scale parameter. |
| Cauchy $(0, \sigma)$ | Cauchy distribution with known location parameter. |
| LN$(\theta, 1)$ | Log- Normal distribution with known scale parameter. |
| LN$(0, \sigma^2)$ | Log- Normal distribution with known location parameter. |
| LL$(\alpha, 1)$ | Log- logistic distribution with known shape parameter. |
| LL$(0, \beta)$ | Log- logistic distribution with known scale parameter. |
| MLE | Maximum likelihood estimator. |

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# Abstract

**Omari, Dareen Atif. Characterizing the Best Ranked Set Sampling Scheme via the Fisher Information. Master of Science Thesis, Department of Statistics, Yarmouk University, 2009 (Supervisor: Dr. Moh'd Alodat).**

In ranked set sampling with set size equal to $m$, there are $m^m$ different possible sampling schemes. If we assume that the family of parametric probability distributions satisfy some suitable regularity conditions, then at least one of these sampling scheme should maximize the amount of Fisher information collected about the parameter of interest. The aim of this thesis is to characterize the best sampling scheme via the Fisher information numbers for several parametric families.

Moreover, we use these best sampling schemes to obtain unbiased estimators for the parameters of interest. Also we compare them to their counter parts under the McIntyre's ranked set sampling scheme. In case of no unbiased estimators, under best sampling scheme, we obtain the MLE's for the parameters of interest. We show that the asymptotic relative efficiencies of the MLE's under best sampling scheme with respect to their McIntyre's counterparts is greater than or equal to one.

# الملخص

**عمري، دارين عاطف. تمييز أفضل مخطط في العينات المرتبة من خلال معلومات فيشر. رسالة ماجستير في العلوم، قسم الاحصاء، جامعة اليرموك 2009.**

**(المشرف: الدكتور محمد العودات).**

في المعاينة المرتبة حيث حجم المجموعة يساوي $m$،هناك $m^m$ من طرائق المعاينة المرتبة المختلفة. اذا افترضنا أن عائلة من التوزيعات الاحتمالية المعلمية تحقق بعض شروط الانتظام المناسبة، فانه يوجد على الأقل واحد من هذه الطرائق تعظم حجم معلومات فيشر التي تحصد حول المعلمة قيد الاهتمام. هدف هذه الأطروحة هو تمييز أفضل مخطط للمعاينة المرتبة بمساعدة معلومات فيشر لمجموعة من التوزيعات الاحتمالية المعلمية.

علاوة على ذلك، سوف نستخدم هذه الطرائق المثلى للحصول على تقديرات احصائية غير منحازة، للمعالم قيد الاهتمام. كذلك سوف نقارن هذه التقديرات مع مثيلاتها في أسلوب ماكنتير. في حالة عدم وجود تقديرات غير منحازة تحت أفضل أسلوب للمعاينة ، سوف نلجأ الى ايجاد تقدير الامكان الأعظم.

سوف نبرهن أن الكفاءة النسبية التقاربية لتقدير الامكان الأعظم في حالة أفضل مخطط بالنسبة لنظيره في حالة أسلوب ماكنتير، هي أكبر من أو تساوي واحد.

## Chapter One

# Introduction

## 1.1 Introduction

The idea of Ranked Set Sampling ( RSS ) was first introduced by McIntyre (1952). He was aiming to increase the efficiency of the sample mean as an estimator for the population mean. The RSS is widely used in situation where the member of a Simple Random Sample (SRS) can be conveniently ordered by a visual inspection or by other crude methods, and where the exact quantification of the sample is expensive in time or effort. To describe the procedure of selecting a RSS, we consider the following notation. Let $X_1, \dots, X_m$ denote a random sample of size $m$ from a distribution function *F(x)*. We use $X_{1:m}, \dots, X_{m:m}$ to denote the order statistics of $X_1, \dots, X_m$. An unbiased estimator for $\mu$ is $\bar{X}_{SRS}$, the sample mean of $X_1, \dots, X_m$. Instead of taking a single SRS, McIntyre proposed taking $m$ independent random samples each of size $m$ but making only one measurement from each sample. He specifically assumed that once a sample has been selected, it is possible to rank its units to measure $X_{1:m}$, the smallest unit in the first sample, $X_{2:m}$ the $2^{nd}$smallest unit from the second sample, and finally $X_{m:m}$ the largest unit from the $m^{th}$ sample. This procedure could be repeated $k$ times (cycles) to obtain more observations. Then $\bar{X}_{RSS}$, the mean of the measured units, i.e., $\bar{X}_{RSS} = \frac{1}{m}\sum_{j=1}^{m} X_{j:m}$ is used to estimate $\mu$.

Takahasi and Wakimoto (1968) showed that $\bar{X}_{RSS}$ is the minimum variance unbiased estimator for $\mu$ with variance $\text{Var}(\bar{X}_{RSS}) = \frac{1}{m^2}\sum_{j=1}^{m} \sigma_{j:m}^2$, where $\sigma_{j:m}^2 = Var(X_{j:m})$. In RSS *a set of $m^2$ sampling units is selected, but only $m$ of them are chosen for actual* quantification. Hence the quantified measurements are independent but not identically distributed. The idea of McIntyre was modified by several authors to produce different sampling schemes which are easier to the experimenter to do the visual ranking see (Muttlak, 1998; Al-odat and Al-Saleh, 2001; Samawi et al., 1996 ; Al-Saleh and Al-Omari, 2002).

Muttlak (1997) proposed the use of Median Ranked Set Sampling (MRSS). According to his proposal, only the median, $M_i$, of each random sample in the RSS scheme is measured. The mean of the medians, $M_i$, ( i = 1, 2,..., n), is a natural estimator of the population location parameter based on MRSS.

Barabesi and El-Shaarawi *(2001) obtained the best ranked set sampling scheme under the* following two distributions:

**a)** Uniform distribution:

For the uniform (0, $\theta$) distribution, they found that the variance of the sample mean obtained via quantifying only the $j^{th}$ order statistic decreases as $j$ increases with the minimum at $j=n$. Hence the optimal selection is the largest order statistic.

**b)** Exponential distribution:

Under the standard exponential distribution, they found the efficiency of SRS, RSS and modified RSS based on only the smallest unit, the second smallest unit, up to the largest unit, denoted by $R_1, R_2, \ldots, R_n$, for samples of size 1 to 6. They found that the choice of $R_1$ for each sample has the same efficiency as SRS, but starting with $n$=2, 3,…, 6 and $R_2, \ldots, R_6$, the modified RSS has a smaller variance than the usual RSS. He showed that the most efficient design to estimate the population mean is the one based on the largest unit in each sample. For more information about modification of RSS see Muttlak(1997), Muttlak and Abu-Dayyeh (2004 ), Al-odat and Al-Saleh (2001), Alodat and Al-Sagheer (2007) and Lam et al.,(1994).

Lavine (1999) evaluated the RSS from a Bayesian point of view. He used the Kullback-Leibler information measure to compare the RSS with a SRS. The following theorem summarizes his results:

**Theorem.** (Lavine, 1999) Let $X_1, X_2 \ldots, X_n$ be a random sample of size $n$ from a pdf $f(x;\theta)$. For any prior distribution $P$ and a sample size $n$, there exists a collection of ranks $r_1, r_2, \ldots, r_m$ such that the expected information in $X_{r_1:n}, \ldots, X_{r_m:n}$ is greater than or equal to the expected information in a simple random sample of size $n$.

Al-Saleh (2002) showed that under some reasonable conditions, any reasonable estimator of any parameter of a population based on a SRS, can be always dominated, in terms of mean squares error, by the corresponding estimator using some RSS plans.

Lavine (1999) and Al-Saleh (2002) show that any SRS can be always dominated by a RSS plan. In fact their results do not tell us what are the best choices of $m$ and $r_1, r_2, \dots, r_m$.

Lam et al., (1994) considered the estimation of the two- parameters exponential distribution

$$f(x;\ \theta, \sigma) = \frac{1}{\sigma} exp\left(\frac{-(x-\theta)}{\sigma}\right), x \geq \theta, \sigma > 0$$

using RSS. Also, they gave several modifications of the RSS. Using these modifications, they obtained BLUE's (Best Linear Unbiased Estimators) for the scale and the location parameters and showed that they have smaller variances in a comparison with their SRS counterparts. The RSS was proposed as a nonparametric method to estimate the mean of the distribution function (McIntyre, 1952; stokes et al., 1988). However, RSS was used as a nonparametric method to estimate several population characteristics, (Lam et al., 1994; Alodat and Al-Sagheer, 2007; Alodat et al., 2009; and Alodat et al., 2009). In this case, it is reasonable to speak about the Fisher information in RSS.

## 1.2 Fisher information

A way of measuring the amount of information that an observable random variable $X$ carries about an unknown parameter $\theta$, upon which the likelihood function of $\theta$ depends, is the Fisher information number. For a single parameter $\theta$, Lehmann and Casella (1998), defined the Fisher information in $X$ about $\theta$ as follows

$$I_X(\theta) = E\left(\frac{\partial log f(X;\ \theta)}{\partial \theta}\right)^2$$

where $f(x;\ \theta)$ is the pdf of $X$, which is assumed to satisfy some regularity conditions.

The amount of Fisher information that a set of ordered statistics contain about a parameter of interest has been considered by several authors. Zheng and Gastwirth (2002) examined the Fisher information in two symmetric fractions of ordered statistics about the scale parameter of a symmetric distribution. Their results show that the tails of an ordered sample contains more information about the scale parameter of a symmetric distribution.

Park and Zheng (2004) derived a necessary and sufficient condition under which two distributions have equal Fisher information in any order statistic. Nagaraja and Abo-Eleneen (2003) gave a survey of results on Fisher information contained in ordered statistics.

Stokes (1995) showed that the Fisher information in a RSS is greater than the Fisher information in a SRS. According to Lavine (1999), we may optimize the information in our RSS scheme by choosing $X_{i_o:n}$ in each set, where $i_o = \arg max_{i=1}^{n} I_{X_{i:n}}(\theta)$.

## 1.3 Thesis objectives

It can be noted that there are $m^m$ different RSS sampling schemes and the McIntyre's RSS is one of these schemes. Till this time, there is no detailed study in the literature, dealing with characterizing the best RSS sampling scheme via the Fisher information and other types of information measures such as Kullback-Leibler and Shannon entropy. In this thesis, we consider the problem of characterizing the best RSS scheme via the Fisher information for several probability distributions.

The rest of this thesis is organized as follows. In chapter two, we derive the mathematical formulas for Fisher information in order statistics for several parametric families. Also, we calculate the amount of information in order statistics based on these formulas. Finally, we use the amount of information to characterize the best sampling schemes.

In chapter three, we employ the best sampling scheme that we obtain in chapter two to derive estimators for the parameters of interest. Also we compare these estimators to their counterparts under the McIntyre sampling scheme via their efficiency. In chapter four, we report our conclusions as well as the future work related to this thesis.

# Chapter Two

## Fisher Information in Order Statistics

### 2.1 Introduction

In this chapter, we obtain the Fisher information in order statistics for several parametric families. Moreover, the amount of information is used to characterize the RSS schemes which collect the largest amount of Fisher information about the parameter of interest.

**Definition 2.1**

Consider $m^2$ sampling units $X_{ij}, i, j = 1, \dots, m$, chosen from a population of interest. By $m$-tuple $(i_1, i_2, \dots, i_m)$, $i_j \in \{1, 2, \dots, m\}$, we mean a modified RSS scheme which chooses, for actual quantification, the $i_1^{th}$ smallest unit from the first set, the $i_2^{th}$ smallest unit from the second set,…, and the $i_m^{th}$ smallest unit from the last set.

To illustrate the above definition, we consider the following example.

**Example**

(1) If $i_1 = 1, i_2 = 2, \dots, i_m = m$, then (1, 2,…, $m$) denotes the McIntyre's RSS scheme.

(2) If $m$=2$k$-1, then ($k$, $k$,…, $k$) denotes the median RSS scheme.

(3) If $m$=2$k$, then (1, 1, …, 1, $m$, …, $m$) denotes the extreme RSS scheme.

### 2.2 Location distributions

Let $X_1, X_2, \dots, X_m$ be iid with $f(x;\ \theta) = f(x - \theta)$. The pdf of $Y_{j:m}$ is

$$f_{Y_{j:m}}(y) = C_{j,m} F^{j-1}(y - \theta)\left(1 - F(y - \theta)\right)^{m-j} f(y - \theta).$$

Where $C_{j,m} = \frac{m!}{(j-1)!(m-j)!}$. The log likelihood function is

$$f^* = logC_{j,m} + (j-1)logF(y-\theta) + (m-j)\,log\big(1-F(y-\theta)\big) + logf(y-\theta).$$

Taking the derivation with respect to $\theta$ we get

$$\frac{\partial f^*}{\partial \theta} = -(j-1)\frac{f(y-\theta)}{F(y-\theta)} + (m-j)\frac{f(y-\theta)}{1-F(y-\theta)} - \frac{f'(y-\theta)}{f(y-\theta)}.$$

Hence

$$I_{Y_{j:m}}(\theta) = E\left(-(j-1)\frac{f(Y_{j:m}-\theta)}{F(Y_{j:m}-\theta)} + (m-j)\frac{f(Y_{j:m}-\theta)}{1-F(Y_{j:m}-\theta)} - \frac{f'(Y_{j:m}-\theta)}{f(Y_{j:m}-\theta)}\right)^2.$$

Using the transformation $v = Y_{j:m} - \theta$, we simplify $I_{Y_{j:m}}(\theta)$ to

$$I_{Y_{j:m}}(\theta) = C_{j,m}\int_{-\infty}^{\infty}\left((j-1)\frac{f(v)}{F(v)} - (m-j)\frac{f(v)}{1-F(v)} + \frac{f'(v)}{f(v)}\right)^2 F^{j-1}(v)\times$$

$$\big(1-F(v)\big)^{m-j} f(v)dv.$$

We note that $I_{Y_{j:m}}(\theta)$ is free of $\theta$.

In what follows, we calculate the values of $I_{Y_{j:m}}(\theta), j = 1, \dots, m$, for different location distributions. Al-Saleh (2002) showed that the difficulty in visual ranking is increasing function in $m$. For that reason, we restrict, in this thesis, our calculations on Fisher information only to $m \in \{3, \dots, 10\}$.

**a. $N(\theta, 1)$**

If $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, then we get the normal case. The Fisher information about $\theta$ in $Y_{j:m}$ is

$$I_{Y_{j:m}}(\theta) = C_{j,m}\int_{-\infty}^{\infty}\left((j-1)\frac{\phi(v)}{\Phi(v)} - (m-j)\frac{\phi(v)}{\Phi(-v)} - v\right)^2 \Phi^{j-1}(v)\Phi^{m-j}(-v)\phi(v)dv.$$

Table 2.1, shows the values of $I_{Y_{j:m}}(\theta)$ for different $j$ and $m$.

Table 2.1 The values of $I_{Y_{j:m}}(\theta)$ for $N(\theta, 1)$ .

| $m$ | $j$ | $I_{Y_{j:m}}(\theta)$ | $m$ | $j$ | $I_{Y_{j:m}}(\theta)$ | $m$ | $j$ | $I_{Y_{j:m}}(\theta)$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 1.82699 | 7 | 1 | 2.70843 | 9 | 4 | 5.87008 |
| | 2 | 2.22924 | | 2 | 3.9448 | | 5 | 6.02095 |
| | 3 | 1.82699 | | 3 | 4.56189 | | 6 | 5.87008 |
| 4 | 1 | 2.10266 | | 4 | 4.75236 | | 7 | 5.39522 |
| | 2 | 2.78057 | | 5 | 4.56189 | | 8 | 4.51451 |
| | 3 | 2.78057 | | 6 | 3.9448 | | 9 | 3.00908 |
| | 4 | 2.10266 | | 7 | 2.70843 | | | |
| 5 | 1 | 2.33337 | 8 | 1 | 2.86605 | 10 | 1 | 3.14013 |
| | 2 | 3.22855 | | 2 | 4.24371 | | 2 | 4.76246 |
| | 3 | 3.48692 | | 3 | 5.00056 | | 3 | 5.75503 |
| | 4 | 3.22855 | | 4 | 5.34475 | | 4 | 6.34463 |
| | 5 | 2.33337 | | 5 | 5.34475 | | 5 | 6.62196 |
| 6 | 1 | 2.53261 | | 6 | 5.00056 | | 6 | 6.62196 |
| | 2 | 3.61024 | | 7 | 4.24371 | | 7 | 6.34463 |
| | 3 | 4.06522 | | 8 | 2.86605 | | 8 | 5.75503 |
| | 4 | 4.06522 | 9 | 1 | 3.00908 | | 9 | 4.76246 |
| | 5 | 3.61024 | | 2 | 4.51451 | | 10 | 3.14013 |
| | 6 | 2.53261 | | 3 | 5.39522 | | | |

From Table 2.1, we note that the amount of information about $\theta$, collected by $Y_{j:m}$ , attains its maximum at the median.

**b. Logistic $(\theta, 1)$**

If $f(x) = \frac{e^x}{(1-e^{-x})^2}$ , then we get the logistic case. The Fisher information about $\theta$ in $Y_{j:m}$ is

$$I_{Y_{j:m}}(\theta) = C_{j,m} \int_{-\infty}^{\infty} \left( m - j + 1 - \frac{(m+1)}{(1+e^v)} \right)^2 \frac{e^{-vr}}{(1+e^{-v})^{m+1}} dv.$$

Table 2.2 shows that values of $I_{Y_{j:m}}(\theta)$ for Logistic $(\theta, 1)$ for different $j$ and $m$.

Table 2.2 The values of $I_{Y_{j:m}}(\theta)$ for Logistic $(\theta, 1)$ .

| $m$ | $j$ | $I_{Y_{j:m}}(\theta)$ | $m$ | $j$ | $I_{Y_{j:m}}(\theta)$ | $m$ | $j$ | $I_{Y_{j:m}}(\theta)$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 0.6 | 7 | 1 | 0.777778 | 9 | 4 | 2.18182 |
| | 2 | 0.8 | | 2 | 1.33333 | | 5 | 2.27273 |
| | 3 | 0.6 | | 3 | 1.66667 | | 6 | 2.18182 |
| 4 | 1 | 0.666667 | | 4 | 1.77778 | | 7 | 1.90909 |
| | 2 | 1 | | 5 | 1.66667 | | 8 | 1.45455 |
| | 3 | 1 | | 6 | 1.33333 | | 9 | 0.818182 |
| | 4 | 0.666667 | | 7 | 0.777778 | | | |
| 5 | 1 | 0.714286 | 8 | 1 | 0.8 | 10 | 1 | 0.8333333 |
| | 2 | 1.14286 | | 2 | 1.4 | | 2 | 1.5 |
| | 3 | 1.28571 | | 3 | 1.8 | | 3 | 2 |
| | 4 | 1.14286 | | 4 | 2 | | 4 | 2.33333 |
| | 5 | 0.714286 | | 5 | 2 | | 5 | 2.5 |
| 6 | 1 | 0.75 | | 6 | 1.8 | | 6 | 2.5 |
| | 2 | 1.25 | | 7 | 1.4 | | 7 | 2.33333 |
| | 3 | 1.5 | | 8 | 0.8 | | 8 | 2 |
| | 4 | 1.5 | 9 | 1 | 0.818182 | | 9 | 1.5 |
| | 5 | 1.25 | | 2 | 1.45455 | | 10 | 0.833333 |
| | 6 | 0.75 | | 3 | 1.90909 | | | |

From Table 2.2, we note that the amount of information about $\theta$, collected by $Y_{j:m}$ , attains its maximum at the median.

**c. Cauchy $(\theta, 1)$**

If $f(x) = \frac{1}{\pi(1+x^2)}$ , then we get the Cauchy case. The Fisher information about $\theta$ in $Y_{j:m}$ is

$$I_{Y_{j:m}}(\theta) = \int_{-\infty}^{\infty} \left( \frac{-\frac{(j-1)}{(1+v^2)}}{\left(tan^{-1}(v) + \frac{\pi}{2}\right)} + \frac{\frac{(m-j)}{(1+v^2)}}{\left(\frac{\pi}{2} - tan^{-1}\, v\right)} + \frac{2v}{(1+v^2)} \right)^2 \times$$

$$\frac{C_{j,m}}{\pi(1+v^2)} \left( \frac{1}{\pi} tan^{-1}(v) + \frac{1}{2} \right)^{j-1} \left( \frac{1}{2} - \frac{1}{\pi} tan^{-1}(v) \right)^{m-j} dv.$$

The values of $I_{Y_{j:m}}(\theta)$ for Cauchy $(\theta, 1)$ are presented in Table 2.3 for different $j$ and $m$.

Table 2.3 The values of $I_{Y_{j:m}}(\theta)$ for Cauchy $(\theta, 1)$.

| $m$ | $j$ | $I_{Y_{j:n}}(\theta)$ | $m$ | $j$ | $I_{Y_{j:n}}(\theta)$ | $m$ | $j$ | $I_{Y_{j:n}}(\theta)$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 0.70895 | 7 | 1 | 0.53246 | | 4 | 2.88880 |
| | 2 | 1.09064 | | 2 | 1.38542 | | 5 | 3.16164 |
| | 3 | 0.70895 | | 3 | 2.14191 | | 6 | 2.88880 |
| 4 | 1 | 0.68994 | | 4 | 2.44042 | | 7 | 2.16392 |
| | 2 | 1.31861 | | 5 | 2.14191 | | 8 | 1.23769 |
| | 3 | 1.31861 | | 6 | 1.38542 | | 9 | 0.43023 |
| | 4 | 0.68994 | | 7 | 0.53246 | | | |
| 5 | 1 | 0.64422 | 8 | 1 | 0.47903 | 10 | 1 | 0.38654 |
| | 2 | 1.41396 | | 2 | 1.31823 | | 2 | 1.15257 |
| | 3 | 1.74552 | | 3 | 2.18418 | | 3 | 2.10331 |
| | 4 | 1.41396 | | 4 | 2.72524 | | 4 | 2.95917 |
| | 5 | 0.64422 | | 5 | 2.72524 | | 5 | 3.46274 |
| 6 | 1 | 0.58887 | | 6 | 2.18418 | | 6 | 3.46274 |
| | 2 | 1.42487 | | 7 | 1.31823 | | 7 | 2.95917 |
| | 3 | 2.00768 | | 8 | 0.47903 | | 8 | 2.10331 |
| | 4 | 2.00768 | 9 | 1 | 0.43023 | | 9 | 1.15257 |
| | 5 | 1.42487 | | 2 | 1.23769 | | 10 | 0.38654 |
| | 6 | 0.58887 | | 3 | 2.16392 | | | |

From Table 2.3, we note that the amount of information about $\theta$, collected by $Y_{j:m}$ , attains its maximum at the median for all $m$.

**d. LN $(\theta, 1)$**

In this case we get the Fisher information about $\theta$ in $Y_{j:m}$ as the same of the Fisher information about $\theta$ in $Y_{j:m}$ for N $(\theta, 1)$. Calculations show that $I_{Y_{j:m}}(\theta)$ is the case as for normal distribution obtained in Table 2.1.

## 2.2 Scale distribution

Let $X_1, \dots, X_m$ be iid $f(x;\ \sigma) = f\left(\frac{x}{\sigma}\right)\frac{1}{\sigma}$ , $\sigma > 0$. The pdf of $Y_{j:m}$ is

$$f_{Y_{j:m}}(y;\ \sigma) = \frac{C_{j,m}}{\sigma} F^{j-1}\left(\frac{y}{\sigma}\right)\left(1 - F\left(\frac{y}{\sigma}\right)\right)^{m-j} f\left(\frac{y}{\sigma}\right).$$

The log likelihood is

$$f^* = \log C_{j,m} + (j-1)\log F\left(\frac{y}{\sigma}\right) + (m-j)\log\left(1 - F\left(\frac{y}{\sigma}\right)\right) + \log f\left(\frac{y}{\sigma}\right) - \log\sigma.$$

Taking the derivative with respect to $\sigma$ we get

$$\frac{\partial f^*}{\partial\sigma} = \frac{-y(j-1)}{\sigma^2}\frac{f(y/\sigma)}{F(y/\sigma)} + \frac{y(m-j)}{\sigma^2}\frac{f(y/\sigma)}{1-F(y/\sigma)} - \frac{y}{\sigma^2}\frac{f'(y/\sigma)}{f(y/\sigma)} - \frac{1}{\sigma}.$$

Hence

$$I_{Y_{j:m}}(\sigma) = E\left(\frac{-y(j-1)}{\sigma^2}\frac{f(Y_{j:m}/\sigma)}{F(Y_{j:m}/\sigma)} + \frac{y(m-j)}{\sigma^2}\frac{f(Y_{j:m}/\sigma)}{1-F(Y_{j:m}/\sigma)} - \frac{y}{\sigma^2}\frac{f'(Y_{j:m}/\sigma)}{f(Y_{j:m}/\sigma)} - \frac{1}{\sigma}\right)^2.$$

Using the transformation $v = Y_{j:m}/\sigma$, we simplify $I_{Y_{j:m}}(\sigma)$ to

$$I_{Y_{j:m}}(\sigma) = \frac{C_{j,m}}{\sigma^2}\int_{-\infty}^{\infty}\left(-(j-1)\frac{vf(v)}{F(v)} + (m-j)\frac{vf(v)}{1-F(v)} - \frac{vf'(v)}{f(v)} - 1\right)^2 F^{j-1}(v)\times$$

$$(1-F(v))^{m-j}f(v)dv.$$

Note that maximizing $I_{Y_{j:m}}(\sigma)$ with respect to $j$ is equivalent to maximizing $\sigma^2 I_{Y_{j:m}}(\sigma)$ with respect to $j$.

In what follows, we calculate the values of $\sigma^2 I_{Y_{j:m}}(\sigma), j = 1,\dots,m, m = 3,\dots,10,$ for different scale families.

**a. Exp (θ)**

If $f(x) = e^{-x}$, then we get the special case. The Fisher information about $\theta$ in $Y_{j:m}$ is

$$\theta^2 I_{Y_{j:m}}(\theta) = C_{j,m}\int_0^{\infty}\left(\frac{-(j-1)ve^{-v}}{(1-e^{-v})} + (m-j+1)v - 1\right)^2 (1-e^{-v})^{j-1}e^{-(m-j+1)v}\,dv$$

It can be noted that $Y_{1:m} \sim \exp\left(\frac{\theta}{m}\right)$. So

$$I_{Y_{1:m}}(\theta) = \frac{1}{\theta^2}.$$

To find the value of $j$ which maximizes $I_{Y_{j:m}}(\theta)$ as a function of $j$, it is equivalent to maximize $\theta^2 I_{Y_{j:m}}(\theta)$ for $j$. Let $j_0 = arg\ max_{j=1}^{m} I_{Y_{j:m}}(\theta)$. Since it is difficult to find $j_0$ theoretically, then we find its values for different values of $m$. Table 2.4 shows the values of $\theta^2 I_{Y_{j:m}}(\theta)$ for different values of $m$ and $j$.

Table 2.4 The values of $\theta^2 I_{Y_{j:m}}(\theta)$ for Exp($\theta$) .

| $m$ | $j$ | $\theta^2 I_{Y_{j:m}}(\theta)$ | $m$ | $j$ | $\theta^2 I_{Y_{j:m}}(\theta)$ | $m$ | $j$ | $\theta^2 I_{Y_{j:m}}(\theta)$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 1 | 7 | 1 | 1 | 9 | 4 | 3.91008 |
| | 2 | 1.92468 | | 2 | 1.98828 | | 5 | 4.78933 |
| | 3 | 2.5 | | 3 | 2.94444 | | 6 | 5.55788 |
| 4 | 1 | 1 | | 4 | 3.83111 | | 7 | 6.12294 |
| | 2 | 1.96048 | | 5 | 4.57389 | | 8 | 6.28179 |
| | 3 | 2.77778 | | 6 | 5.00361 | | 9 | 5.4723 |
| | 4 | 3.11111 | | 7 | 4.63144 | | | |
| 5 | 1 | 1 | 8 | 1 | 1 | 10 | 1 | 1 |
| | 2 | 1.97579 | | 2 | 1.99118 | | 2 | 1.99449 |
| | 3 | 2.875 | | 3 | 2.95918 | | 3 | 2.97531 |
| | 4 | 3.56944 | | 4 | 3.87982 | | 4 | 3.93017 |
| | 5 | 3.66204 | | 5 | 4.70987 | | 5 | 4.83992 |
| 6 | 1 | 1 | | 6 | 5.36537 | | 6 | 5.67343 |
| | 2 | 1.98369 | | 7 | 5.65963 | | 7 | 6.37702 |
| | 3 | 2.92 | | 8 | 5.06532 | | 8 | 6.84973 |
| | 4 | 3.745 | 9 | 1 | 1 | | 9 | 6.87407 |
| | 5 | 4.30889 | | 2 | 1.99312 | | 10 | 5.85612 |
| | 6 | 4.16583 | | 3 | 2.96875 | | | |

From Table 2.4, we note that the amount of information about $\theta$, collected by $Y_{j:m}$ , attains its maximum at $j=m$ if $3 \le m \le 5$ and at $j=m-1$ if $6 \le m \le 10$.

**b. N (0, $\sigma^2$)**

If $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$, then we get the special case. The Fisher information about $\sigma$ in $Y_{j:m}$ is

$$\sigma^2 I_{Y_{j:m}}(\sigma) = C_{j,m} \int_{-\infty}^{\infty} \left(Z^*(Y)\right)^2 \Phi^{j-1}(v)\, \Phi^{m-j}(-v)\, \phi(v)\ dv,$$

where,

$$Z^*(Y) = \frac{-(j-1)v\phi(v)}{\Phi(v)} + \frac{(m-j)v\phi(-v)}{\Phi(-v)} + v^2 - 1.$$

The values of $\sigma^2 I_{Y_{j:m}}(\theta)$ for $N(0,\sigma^2)$ are presented in Table 2.5 for different $j$ and $m$.

Table 2.5 The values of $\sigma^2 I_{Y_{j:m}}(\sigma)$ for $N(0,\sigma^2)$ .

| $m$ | $j$ | $\sigma^2 I_{Y_{j:m}}(\theta)$ | $m$ | $j$ | $\sigma^2 I_{Y_{j:m}}(\theta)$ | $m$ | $j$ | $\sigma^2 I_{Y_{j:m}}(\theta)$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 2.82699 | 7 | 1 | 5.47077 | 9 | 4 | 2.35778 |
| | 2 | 1.96644 | | 2 | 3.77183 | | 5 | 1.97732 |
| | 3 | 2.82699 | | 3 | 2.4421 | | 6 | 2.35778 |
| 4 | 1 | 3.47021 | | 4 | 1.97356 | | 7 | 3.46856 |
| | 2 | 2.15021 | | 5 | 2.4421 | | 8 | 5.15627 |
| | 3 | 2.15021 | | 6 | 3.77183 | | 9 | 6.75126 |
| | 4 | 3.47021 | | 7 | 5.47077 | | | |
| 5 | 1 | 4.1389 | 8 | 1 | 6.11879 | 10 | 1 | 7.36749 |
| | 2 | 2.57729 | | 2 | 4.45129 | | 2 | 5.87502 |
| | 3 | 1.96904 | | 3 | 2.91076 | | 3 | 4.08797 |
| | 4 | 2.57729 | | 4 | 2.08112 | | 4 | 2.7566 |
| | 5 | 4.1389 | | 5 | 2.08112 | | 5 | 2.06608 |
| 6 | 1 | 4.80923 | | 6 | 2.91076 | | 6 | 2.06608 |
| | 2 | 3.13666 | | 7 | 4.45129 | | 7 | 2.7566 |
| | 3 | 2.10517 | | 8 | 6.11879 | | 8 | 4.08797 |
| | 4 | 2.10517 | 9 | 1 | 6.75126 | | 9 | 5.87502 |
| | 5 | 3.13666 | | 2 | 5.15627 | | 10 | 7.36749 |
| | 6 | 4.80923 | | 3 | 3.46856 | | | |

From Table 2.5, we note that the amount of information about $\sigma$, collected by $Y_{j:m}$ , attains its maximum at $j = 1$ or $m$.

**c. Logistic (0, σ)**

If $f(x) = \frac{1}{\sigma} \frac{e^{-\left(\frac{x}{\sigma}\right)}}{\left(1+e^{-\left(\frac{x}{\sigma}\right)}\right)^2}$ , then we get the special case. The Fisher information about $\sigma$ in $Y_{j:m}$ is

$$\sigma^2 I_{Y_{j:m}}(\sigma) = C_{j,m} \int_{-\infty}^{\infty} \left(-1 + v(m-j+1) - \frac{(m+1)v}{(1+e^{v})}\right)^2 \frac{e^{-vr}}{(1+e^{-v})^{m+1}} dv.$$

The values of $\sigma^2 I_{Y_{j:m}}(\sigma)$ for Logistic $(0, \sigma)$ are presented in Table 2.6.

Table 2.6 The values of $\sigma^2 I_{Y_{j:m}}(\sigma)$ for Logistic $(0, \sigma)$ .

| $m$ | $j$ | $\sigma^2 I_{Y_{j:m}}(\sigma)$ | $m$ | $j$ | $\sigma^2 I_{Y_{j:m}}(\sigma)$ | $m$ | $j$ | $\sigma^2 I_{Y_{j:m}}(\sigma)$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 1.97392 | 7 | 1 | 3.57854 | | 4 | 2.11123 |
| | 2 | 1.63189 | | 2 | 2.93464 | | 5 | 1.8242 |
| | 3 | 1.97392 | | 3 | 2.11274 | | 6 | 2.11123 |
| 4 | 1 | 2.35991 | | 4 | 1.78693 | | 7 | 2.89732 |
| | 2 | 1.78987 | | 5 | 2.11274 | | 8 | 3.90272 |
| | 3 | 1.78987 | | 6 | 2.93464 | | 9 | 4.35064 |
| | 4 | 2.35991 | | 7 | 3.57854 | | | |
| 5 | 1 | 2.76657 | 8 | 1 | 3.97078 | 10 | 1 | 4.71752 |
| | 2 | 2.09318 | | 2 | 3.41137 | | 2 | 4.39939 |
| | 3 | 1.72983 | | 3 | 2.47176 | | 3 | 3.36743 |
| | 4 | 2.09318 | | 4 | 1.88529 | | 4 | 2.43284 |
| | 5 | 2.76657 | | 5 | 1.88529 | | 5 | 1.90661 |
| 6 | 1 | 3.17573 | | 6 | 2.47176 | | 6 | 1.90661 |
| | 2 | 2.48734 | | 7 | 3.41137 | | 7 | 2.43284 |
| | 3 | 1.85147 | | 8 | 3.97078 | | 8 | 3.36743 |
| | 4 | 1.85147 | 9 | 1 | 4.35064 | | 9 | 4.39939 |
| | 5 | 2.48734 | | 2 | 3.90272 | | 10 | 4.71752 |
| | 6 | 3.17573 | | 3 | 2.89732 | | | |

From Table 2.6, we note that the amount of information a bout $\sigma$, collected by $Y_{j:m}$ , attains its maximum at $j = 1$ or $m$.

**d. Cauchy $(0, \sigma)$**

If $f(x) = \frac{1}{\pi\sigma} \frac{1}{1+\left(\frac{x}{\sigma}\right)^2}$ , then we get the special case. The Fisher information about $\sigma$ in $Y_{j:m}$ is

$$\sigma^2 I_{Y_{j:m}}(\sigma) = \int_{-\infty}^{\infty} \left( \frac{-\frac{v(j-1)}{(1+v^2)}}{tan^{-1} v + \frac{\pi}{2}} + \frac{\frac{v(m-j)}{(1+v^2)}}{\frac{\pi}{2} - tan^{-1} v} - 1 + \frac{2v^2}{(1+v^2)} \right)^2 \times$$

$$\left(\frac{1}{\pi} tan^{-1} v + \frac{1}{2}\right)^{j-1} \frac{C_{j,m}}{\pi(1+v^2)} \left(\frac{1}{2} - \frac{1}{\pi} tan^{-1}(v)\right)^{m-j} dv.$$

The values of $\sigma^2 I_{Y_{j:m}}(\sigma)$ for Cauchy $(0, \sigma)$ are presented in Table 2.7.

Table 2.7 The values of $\sigma^2 I_{Y_{j:m}}(\sigma)$ for Cauchy $(0, \sigma)$.

| $m$ | $j$ | $\sigma^2 I_{Y_{j:m}}(\sigma)$ | $m$ | $j$ | $\sigma^2 I_{Y_{j:m}}(\sigma)$ | $m$ | $j$ | $\sigma^2 I_{Y_{j:m}}(\sigma)$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 0.581536 | 7 | 1 | 0.679664 | 9 | 4 | 1.24559 |
| | 2 | 0.782855 | | 2 | 1.031140 | | 5 | 1.21406 |
| | 3 | 0.581536 | | 3 | 1.108600 | | 6 | 1.24559 |
| 4 | 1 | 0.596172 | | 4 | 1.109950 | | 7 | 1.28287 |
| | 2 | 0.858846 | | 5 | 1.108600 | | 8 | 1.17698 |
| | 3 | 0.858846 | | 6 | 1.031140 | | 9 | 0.735834 |
| | 4 | 0.596172 | | 7 | 0.679664 | | | |
| 5 | 1 | 0.620030 | 8 | 1 | 0.708976 | 10 | 1 | 0.759727 |
| | 2 | 0.909458 | | 2 | 1.10305 | | 2 | 1.24958 |
| | 3 | 0.972904 | | 3 | 1.18911 | | 3 | 1.38576 |
| | 4 | 0.909458 | | 4 | 1.17076 | | 4 | 1.33981 |
| | 5 | 0.620030 | | 5 | 1.17076 | | 5 | 1.26867 |
| 6 | 1 | 0.649182 | | 6 | 1.18911 | | 6 | 1.26867 |
| | 2 | 0.965504 | | 7 | 1.10305 | | 7 | 1.33981 |
| | 3 | 1.041040 | | 8 | 0.708976 | | 8 | 1.38576 |
| | 4 | 1.041040 | 9 | 1 | 0.735834 | | 9 | 1.24958 |
| | 5 | 0.965504 | | 2 | 1.17698 | | 10 | 0.759727 |
| | 6 | 0.649182 | | 3 | 1.28287 | | | |

From Table 2.7, we note that the amount of information about $\sigma$, collected by $Y_{j:m}$ , attains its maximum at the median if $3 \leq m \leq 7$ and at $j$=3 and $j$=$m$-2 if $8 \leq m \leq 10$.

**e. LN$(0, \sigma^2)$**

In this case we get the Fisher information about $\theta$ in $Y_{j:m}$ as the same of the Fisher information about $\sigma$ in $Y_{j:m}$ for N$(0, \sigma^2)$. Calculations show that $\sigma^2 I_{Y_{j:m}}(\sigma)$ is the case as for normal distribution obtained in Table 2.5.

## 2.3. Location- scale family with constant coefficient of variation

In variety of situations, the coefficient of variation (c.v.) of the statistical population is known. Such statistical populations arise in agricultural, biology and sample survey (Sinha, 1983; Alodat et al., 2009). In this section, we assume that the population of interest has a known c.v. equals $c$. Without loss of generality, we may assume that $c$=1. For location-scale family with known c.v. we will use c=1 in our calculation.

Let $X_1, \ldots, X_m$ be iid $f\left(\frac{x-\theta}{\sigma}\right)\frac{1}{\sigma}$, $\sigma > 0,\ -\infty < \theta < \infty$. We assume that $f(x)$ is a differentiable function. Then the mean of $X$ is

$$\mu = E(X) = \int_{-\infty}^{\infty} \frac{x}{\sigma} f\left(\frac{x-\theta}{\sigma}\right) dx .$$

Using the transformation $v = \frac{x-\theta}{\sigma}$, we can write $\mu$ as follows

$$\mu = \int_{-\infty}^{\infty} \left(\frac{\theta + \sigma v}{\sigma}\right) f(v)\sigma\, dv,$$

$$= \theta + \sigma \int_{-\infty}^{\infty} v\, f(v) dv,$$

$$= \theta + \sigma A,$$

where $A = \int_{-\infty}^{\infty} v\, f(v) dv$. Also, the variance of $X$ can be written as

$$var(X) = \int_{-\infty}^{\infty} \frac{x^2}{\sigma} f\left(\frac{x-\theta}{\sigma}\right) dx - \mu^2,$$

$$= \sigma^2 \left( \int_{-\infty}^{\infty} v^2\, f(v) - \left(\int_{-\infty}^{\infty} v\, f(v)\right)^2 \right),$$

$$= \sigma^2 D,$$

where

$$D = \int_{-\infty}^{\infty} v^2\, f(v) - \left(\int_{-\infty}^{\infty} v\, f(v)\right)^2.$$

Note that $D$ depends on the function form $f(v)$.

If $\frac{1}{\gamma}$ is the c.v. of the population of $f$, then

$$\gamma = \frac{\theta + \sigma\ A}{\sqrt{D\ \sigma^2}} = \frac{\theta + \sigma\ A}{\sigma\sqrt{D}},$$

$$= \frac{\theta}{\sigma\sqrt{D}} + \frac{A}{\sqrt{D}}.$$

Note that $\gamma$ is known iff $\frac{\theta}{\sigma}$ is known. Hence $\theta = c^*\sigma$, where

$$c^* = \gamma\sqrt{D} - A.$$

Note that $c^*$ is also known iff $\frac{\theta}{\sigma}$ is known. Since $c^*$ is known, then $f\left(\frac{x-\theta}{\sigma}\right)\frac{1}{\sigma}$ can be written as

$$f(x\,,\sigma) = \frac{1}{\sigma}\,f\left(\frac{x - c^*\,\sigma}{\sigma}\right)$$

$$= \frac{1}{\sigma}\,f\left(\frac{x}{\sigma} - c^*\right).$$

The CDF of $f$ is

$$F(x\,;\,\sigma) = F\left(\frac{x-\theta}{\sigma}\right),$$

$$= F\left(\frac{x - c^*\,\sigma}{\sigma}\right),$$

$$= F\left(\frac{x}{\sigma} - c^*\right).$$

The pdf of $Y_{j:m}$ is

$$g(y\,;\,\sigma) = C_{j,m}\ \left[F\left(\frac{y}{\sigma} - c^*\right)\right]^{j-1}\ \left[1 - F\left(\frac{y}{\sigma} - c^*\right)\right]^{m-j}\ \frac{1}{\sigma}\,f\left(\frac{y}{\sigma} - c^*\right).$$

We find the Fisher information about $\sigma$ contained in $Y_{j:m}$ as follows

$$g^* = \log C_{j,m} + c_1 \log F\left(\frac{y}{\sigma} - c^*\right) + c_2 \log\left[1 - F\left(\frac{y}{\sigma} - c^*\right)\right] + \log f\left(\frac{y}{\sigma} - c^*\right) - \log \sigma,$$

where $g^* = \log g(y;\sigma)$, $c_1 = j - 1$ and $c_2 = m - j$.

Taking the derivative of $g^*$ with respect to $\sigma$, then

$$\frac{\partial g^*}{\partial \sigma} = \frac{c_1 f\left(\frac{y}{\sigma} - c^*\right)}{F\left(\frac{y}{\sigma} - c^*\right)}\left(-\frac{y}{\sigma^2}\right) + \frac{c_2 f\left(\frac{y}{\sigma} - c^*\right)}{1 - F\left(\frac{y}{\sigma} - c^*\right)}\left(\frac{y}{\sigma^2}\right) - \frac{f'\left(\frac{y}{\sigma} - c^*\right)}{f\left(\frac{y}{\sigma} - c^*\right)}\left(\frac{y}{\sigma^2}\right) - \frac{1}{\sigma},$$

$$I_{Y_{j:m}}(\sigma) = E\left(\frac{\partial g^*}{\partial \sigma}\right)^2,$$

$$= \int_{-\infty}^{\infty}\left(\frac{c_1 f\left(\frac{y}{\sigma} - c^*\right)}{F\left(\frac{y}{\sigma} - c^*\right)}\left(-\frac{y}{\sigma^2}\right) + \frac{c_2 f\left(\frac{y}{\sigma} - c^*\right)}{1 - F\left(\frac{y}{\sigma} - c^*\right)}\left(\frac{y}{\sigma^2}\right) - \frac{f'\left(\frac{y}{\sigma} - c^*\right)\left(\frac{y}{\sigma^2}\right)}{f\left(\frac{y}{\sigma} - c^*\right)} - \frac{1}{\sigma}\right)^2 \times$$

$$C_{j,m} F^{j-1}\left(\frac{y}{\sigma} - c^*\right)\left(1 - F\left(\frac{y}{\sigma} - c^*\right)\right)^{m-j} f\left(\frac{y}{\sigma} - c^*\right)\frac{dy}{\sigma}.$$

$$I_{Y_{j:m}}(\sigma) = \int_{-\infty}^{\infty}\left(\frac{-c_1 f(w)}{F(w)}\frac{\sigma(c^* + w)}{\sigma^2} + \frac{c_2 f(w)}{1 - F(w)}\frac{\sigma(c^* + w)}{\sigma^2} - \frac{f'(w)}{f(w)}\frac{\sigma(c^* + w)}{\sigma^2} - \frac{1}{\sigma}\right)^2 \times$$

$$F^{j-1}(w)\left(1 - F(w)\right)^{m-j} f(w)dw,$$

$$= \frac{C_{j,m}}{\sigma^2}\int_{-\infty}^{\infty}\left(\frac{-c_1 f(w)}{F(w)}(c^* + w) + \frac{c_2 f(w)}{1 - F(w)}(c^* + w) - \frac{f'(w)}{f(w)}(c^* + w) - 1\right)^2 \times$$

$$F^{j-1}(w)\left(1 - F(w)\right)^{m-j} f(w)dw,$$

$$= I_{f,j}(w).$$

Note that $I_{Y_{j:m}}(\sigma)$ is a function of $j$, $m$, $\sigma$, and $f$. It is enough to maximize $\sigma^2 I_{Y_{j:m}}(\sigma)$ with respect to $j$ instead of maximizing $I_{Y_{j:m}}(\sigma)$.

In the following, we calculate the values of $\sigma^2 I_{Y_{j:m}}(\sigma)$, $j$=1, ..., $m$, $m$=3, ..., 10, for different location- scale distribution with known c.v.

**a. N(θ, θ²)**

Let $X_1, X_2, \dots, X_m$ iid $N(\theta,\ \theta^2),\ \theta > 0$. The pdf of $X_i$ is

$$f(x;\theta) = \frac{1}{\sqrt{2\pi}}\frac{1}{\theta}e^{-\frac{(x-\theta)^2}{2\theta^2}}, \theta > 0;\ -\infty < x < \infty.$$

The Fisher information about $\theta$ in $Y_{j:m}$ is

$$I_{Y_{j:m}}(\theta) = \frac{C_{j,m}}{\theta^2}\int_{-\infty}^{\infty}\left(v + v^2 - 1 - \frac{(j-1)(v+1)\phi(v)}{\Phi(v)} + \frac{(m-j)\phi(v)(v+1)}{\Phi(-v)}\right)^2 \times$$

$$\Phi^{j-1}(v)\ \Phi^{m-j}(-v)\phi(v)dv.$$

Therefore

$$arg\ max_{j=1}^{m}\ I_{Y_{j:m}}(\theta) = arg\ max_{j=1}^{m}\ \theta^2\ I_{Y_{j:m}}(\theta).$$

The values of $\theta^2 I_{Y_{j:m}}(\theta)$ for normal distributions with known $c$ are presented in Table 2.8. for different $j$ and $m$.

Table 2.8 The values of $\theta^2 I_{Y_{j:m}}(\theta)$ for $N(\theta,\theta^2)$.

| $m$ | $j$ | $\theta^2 I_{Y_{j:m}}(\theta)$ | $m$ | $j$ | $\theta^2 I_{Y_{j:m}}(\theta)$ | $m$ | $j$ | $\theta^2 I_{Y_{j:m}}(\theta)$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 2.11476 | 7 | 1 | 1.96797 | | 4 | 5.22923 |
| | 2 | 4.19568 | | 2 | 2.35841 | | 5 | 7.99827 |
| | 3 | 7.19200 | | 3 | 4.06587 | | 6 | 11.2265 |
| 4 | 1 | 1.97630 | | 4 | 6.72592 | | 7 | 14.5578 |
| | 2 | 3.49021 | | 5 | 9.94211 | | 8 | 17.284 |
| | 3 | 6.37135 | | 6 | 13.0749 | | 9 | 17.398 |
| | 4 | 9.16783 | | 7 | 14.3876 | | | |
| 5 | 1 | 1.92452 | 8 | 1 | 2.03416 | 10 | 1 | 2.21782 |
| | 2 | 2.98160 | | 2 | 2.17796 | | 2 | 1.98397 |
| | 3 | 5.45596 | | 3 | 3.56809 | | 3 | 2.85174 |
| | 4 | 8.63008 | | 4 | 5.91783 | | 4 | 4.64580 |
| | 5 | 11.0180 | | 5 | 8.93391 | | 5 | 7.16423 |
| 6 | 1 | 1.92774 | | 6 | 12.2546 | | 6 | 10.2119 |
| | 2 | 2.61738 | | 7 | 15.2121 | | 7 | 13.5567 |
| | 3 | 4.68684 | | 8 | 15.9323 | | 8 | 16.8342 |
| | 4 | 7.65393 | 9 | 1 | 2.11909 | | 9 | 19.2910 |
| | 5 | 10.8764 | | 2 | 2.05752 | | 10 | 18.7934 |
| | 6 | 12.7535 | | 3 | 3.16972 | | | |

From Table 2.8, we note that the amount of information about $\theta$, collected by $Y_{j:m}$ , attains its maximum at $j = m,\ 3 \le m \le 9$ and at $j = m - 1,\ m = 10$.

**b. Logistic distribution with known c.v.**

Let $X_1, X_2, \dots, X_m$ be iid logistic $(\theta, \sigma)$.

$$f(x) = \frac{1}{\sigma}\frac{e^{-(x-\theta)/\sigma}}{(1+e^{-(x-\theta)/\sigma})^2}, -\infty < x < \infty, -\infty < \theta < \infty, \sigma > 0.$$

$cv = \frac{\pi\sigma}{\theta\sqrt{3}} = c$, say. So we get that $\sigma = \frac{c\theta\sqrt{3}}{\pi}$. Substituting that in $f(x)$ we get

$$f(x) = \frac{\pi}{c\theta\sqrt{3}}\frac{e^{-\frac{(x-\theta)\pi}{c\theta\sqrt{3}}}}{\left(1+e^{-\frac{(x-\theta)\pi}{c\theta\sqrt{3}}}\right)^2}.$$

Then, we get the Fisher information about $\theta$ in $Y_{j:m}$ as

$$I_{Y_{j:m}}(\theta)=\frac{1}{\theta^2}\int_{-\infty}^{\infty}\left(-1+\frac{(v+1)\pi(m-j+1)}{c\sqrt{3}}-\frac{(m+1)\pi(v+1)e^{-\frac{\pi v}{c\sqrt{3}}}}{c\sqrt{3}\left(1+e^{-\frac{\pi v}{c\sqrt{3}}}\right)}\right)^2\times$$

$$\frac{\pi C_{j,m}e^{-\frac{\pi v(m-j+1)}{c\sqrt{3}}}}{c\sqrt{3}\left(1+e^{-\frac{\pi v}{c\sqrt{3}}}\right)^{m+1}}dv.$$

The values of $\theta^2 I_{Y_{j:m}}(\theta)$ for logistic with known $c$ are presented in Table 2.9 for different $j$ and $m$.

Table 2.9 The values of $\theta^2 I_{Y_{j:m}}(\theta)$ for Logistic with $c=1$.

| $m$ | $j$ | $\theta^2 I_{Y_{j:m}}(\theta)$ | $m$ | $j$ | $\theta^2 I_{Y_{j:m}}(\theta)$ | $m$ | $j$ | $\theta^2 I_{Y_{j:m}}(\theta)$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 2.13403 | 7 | 1 | 1.64311 | 9 | 4 | 6.38704 |
| | 2 | 4.26379 | | 2 | 2.72617 | | 5 | 9.30117 |
| | 3 | 5.76163 | | 3 | 4.87516 | | 6 | 12.1912 |
| 4 | 1 | 1.93321 | | 4 | 7.63558 | | 7 | 14.4380 |
| | 2 | 3.87054 | | 5 | 10.3166 | | 8 | 15.1140 |
| | 3 | 6.28894 | | 6 | 11.9161 | | 9 | 12.4707 |
| | 4 | 7.17307 | | 7 | 10.6315 | | | |
| 5 | 1 | 1.79116 | 8 | 1 | 1.61729 | 10 | 1 | 1.62777 |
| | 2 | 3.43463 | | 2 | 2.46696 | | 2 | 2.10274 |
| | 3 | 5.95966 | | 3 | 4.36689 | | 3 | 3.52978 |
| | 4 | 8.27143 | | 4 | 7.01399 | | 4 | 5.79639 |
| | 5 | 8.44175 | | 5 | 9.91607 | | 5 | 8.61979 |
| 6 | 1 | 1.6981 | | 6 | 12.4202 | | 6 | 11.6428 |
| | 2 | 3.04765 | | 7 | 13.5674 | | 7 | 14.422 |
| | 3 | 5.42592 | | 8 | 11.588 | | 8 | 16.3646 |
| | 4 | 8.14662 | 9 | 1 | 1.61387 | | 9 | 16.5656 |
| | 5 | 10.1517 | | 2 | 2.26195 | | 10 | 13.2903 |
| | 6 | 9.58812 | | 3 | 3.91796 | | | |

From Table 2.9, we note that the amount of information about $\theta$, collected by $Y_{j:m}$, attains its maximum at $j=m$ if $3\le m\le 5$ and , $j=m-1$ if $6\le m\le 10$.

## c. Cauchy distribution with known c.v.

If $\frac{\sigma}{\theta} = c$ is known, then

$$f(x) = \frac{1}{c\theta\pi} \frac{1}{1 + \left(\frac{x-\theta}{c\theta}\right)^2}.$$

Then we get the Fisher information about $\theta$ in $Y_{j:m}$ as

$$\theta^2 I_{Y_{j:m}}(\theta) = \int_{-\infty}^{\infty} \left( \frac{-\frac{(j-1)(cv+1)}{\pi c(1+v^2)}}{\frac{1}{\pi}tan^{-1}v + \frac{1}{2}} + \frac{\frac{(m-j)(cv+1)}{\pi c(1+v^2)}}{\frac{1}{2} - \frac{1}{\pi}tan^{-1}v} - 1 + \frac{2v(cv+1)}{c(1+v^2)} \right)^2 \times$$

$$\frac{C_{j,m}\left(\frac{1}{\pi}tan^{-1}v + \frac{1}{2}\right)^{j-1}}{\pi(1+v^2)} \left(\frac{1}{2} - \frac{1}{\pi}tan^{-1}v\right)^{m-j} dv.$$

The values of $\theta^2 I_{Y_{j:m}}(\theta)$ for Cauchy$(\theta, \sigma)$ with known $c$ are presented in Table 2.10 for different $j$ and $m$.

Table 2.10 The values of $\theta^2 I_{Y_{j:m}}(\theta)$ for Cauchy$(\theta, \sigma)$ with $c = 1$.

| $m$ | $j$ | $\theta^2 I_{Y_{j:n}}(\theta)$ | $m$ | $j$ | $\theta^2 I_{Y_{j:n}}(\theta)$ | $m$ | $j$ | $\theta^2 I_{Y_{j:n}}(\theta)$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 1.03645 | 7 | 1 | 0.562966 | 9 | 4 | 3.22897 |
| | 2 | 1.87133 | | 2 | 1.43047 | | 5 | 4.3757 |
| | 3 | 1.50983 | | 3 | 2.52843 | | 6 | 5.03976 |
| 4 | 1 | 0.887419 | | 4 | 3.55036 | | 7 | 4.85185 |
| | 2 | 1.9483 | | 5 | 3.97194 | | 8 | 3.68836 |
| | 3 | 2.40227 | | 6 | 3.38843 | | 9 | 1.77597 |
| | 4 | 1.63963 | | 7 | 1.78757 | | | |
| 5 | 1 | 0.75135 | 8 | 1 | 0.505495 | 10 | 1 | 0.439256 |
| | 2 | 1.82067 | | 2 | 1.25813 | | 2 | 1.00769 |
| | 3 | 2.71824 | | 3 | 2.28843 | | 3 | 1.81124 |
| | 4 | 2.81917 | | 4 | 3.45688 | | 4 | 2.94355 |
| | 5 | 1.72197 | | 5 | 4.3351 | | 5 | 4.21353 |
| 6 | 1 | 0.64334 | | 6 | 4.45712 | | 6 | 5.24929 |
| | 2 | 1.62667 | | 7 | 3.56578 | | 7 | 5.65434 |
| | 3 | 2.70503 | | 8 | 1.78822 | | 8 | 5.16474 |
| | 4 | 3.39204 | 9 | 1 | 0.465745 | | 9 | 3.7676 |
| | 5 | 3.14375 | | 2 | 1.11737 | | 10 | 1.75516 |
| | 6 | 1.76808 | | 3 | 2.04019 | | | |

From Table 2.10, we note that the amount of information about $\theta$, collected by $Y_{j:m}$,

attains its maximum at $j = m - 1,\ 3 \le m \le 5, j = m - 2,\ 6 \le m \le 8$, and

$j = m - 3, m = 9, 10$.

**d. Log- normal with known c.v.**

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(\log x - \theta)^2}, \qquad 0 < x < \infty, -\infty < \theta < \infty.$$

Now, $cv = \sqrt{e^{\sigma^2} - 1} = c$, say. This implies that $\sigma^2 = \log(c^2 + 1)$.

Substituting $\sigma^2$ in $f(x)$ we get

$$f(x) = \frac{1}{x\sqrt{\log(c^2+1)}\sqrt{2\pi}} e^{-\frac{1}{2\log(c^2+1)}(\log x - \theta)^2}.$$

Then we get the Fisher information about $\theta$ in $Y_{j:m}$ as

$$I_{Y_{j:m}}(\theta) = \frac{C_{j,m}}{\sqrt{2\pi}\log(c^2+1)} \int_{-\infty}^{\infty} \left(v - \frac{(j-1)\phi(v)}{\Phi(v)} + \frac{(m-j)\phi(-v)}{\Phi(-v)}\right)^2 \times$$

$$\Phi^{j-1}(v)\Phi^{m-j}(-v)e^{-\frac{v^2}{2}}dv.$$

The values of $I_{Y_{j:m}}(\theta)$ for Log Normal with known $c$ are presented in Table 2.11 for different $j$ and $m$.

Table 2.11 The values of $I_{Y_{j:m}}(\theta)$ for Log Normal with c=1.

| $m$ | $j$ | $I_{Y_{j:m}}(\theta)$ | $m$ | $j$ | $I_{Y_{j:n}}(\theta)$ | $m$ | $j$ | $I_{Y_{j:n}}(\theta)$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 2.63579 | 7 | 1 | 3.90743 | 9 | 4 | 8.46873 |
| | 2 | 3.21612 | | 2 | 5.69115 | | 5 | 8.68639 |
| | 3 | 2.63579 | | 3 | 6.58141 | | 6 | 8.46873 |
| 4 | 1 | 3.03349 | | 4 | 6.85621 | | 7 | 7.78366 |
| | 2 | 4.01151 | | 5 | 6.58141 | | 8 | 6.51307 |
| | 3 | 4.01151 | | 6 | 5.69115 | | 9 | 4.34118 |
| | 4 | 3.03349 | | 7 | 3.90743 | | | |
| 5 | 1 | 3.36634 | 8 | 1 | 4.13483 | 10 | 1 | 4.53025 |
| | 2 | 4.65782 | | 2 | 6.12239 | | 2 | 6.87078 |
| | 3 | 5.03056 | | 3 | 7.21428 | | 3 | 8.30275 |
| | 4 | 4.65782 | | 4 | 7.71084 | | 4 | 9.15336 |
| | 5 | 3.36634 | | 5 | 7.71084 | | 5 | 9.55347 |
| 6 | 1 | 3.65378 | | 6 | 7.21428 | | 6 | 9.55347 |
| | 2 | 5.20848 | | 7 | 6.12239 | | 7 | 9.15336 |
| | 3 | 5.86487 | | 8 | 4.13483 | | 8 | 8.30275 |
| | 4 | 5.86487 | 9 | 1 | 4.34118 | | 9 | 6.87078 |
| | 5 | 5.20848 | | 2 | 6.51307 | | 1 | 4.53025 |
| | 6 | 3.65378 | | 3 | 7.78366 | | | |

From Table 2.11, we note that the amount of information about $\theta$, collected by $Y_{j:m}$, attains its maximum at the median for all $m$.

## 2.5. Other distributions

In this section, we consider other parametric families where it is not easy to reduce their 2-parameters versions to a one- parameter version even if their c.v.'s are known. Also some of them do not satisfy suitable regularity conditions in their 2- parameter version.

**a. Beta($\theta$, 1)**

Let $X_1, X_2, \ldots, X_m$ iid Beta$(\theta, 1)$. The pdf of $Y_{j:m}$ is given by

$$f_{Y_{j:m}}(y) = C_{j,m}\, \theta y^{\theta-1} y^{\theta(j-1)} (1-y^{\theta})^{m-j},$$

$$= C_{j,m}\, \theta y^{\theta j-1} (1-y^{\theta})^{m-j}.$$

The log likelihood is

$$\log f_{Y_{j:m}}(y) = \log C_{j,m} + \log\theta + (\theta j - 1)\log y + (m-j)\log(1-y^{\theta}),$$

Taking the derivative with respect to $\theta$, we get

$$\frac{\partial \log f_{Y_{j:m}}(y;\,\theta)}{\partial\theta} = \frac{1}{\theta} + j\log y - \frac{(m-j)y^{\theta}\log y}{(1-y^{\theta})},$$

The Fisher information about $\theta$ in $Y_{j:m}$ is

$$I_{Y_{j:m}}(\theta) = E\left(\frac{1}{\theta} + j\log y - \frac{(m-j)y^{\theta}\log y}{(1-y^{\theta})}\right)^2,$$

$$= C_{j,m}\int_0^1 \left(\frac{1}{\theta} + j\log y - \frac{(m-j)y^{\theta}\log y}{(1-y^{\theta})}\right)^2 \theta y^{\theta j-1}(1-y^{\theta})^{m-j}dy,$$

Using the transformation $v = y^{\theta}$, we get

$$I_{Y_{j:m}}(\theta) = C_{j,m}\int_0^1 \left(\frac{1}{\theta} + \frac{j}{\theta}\log v - \frac{(m-j)v\log v}{\theta(1-v)}\right)^2 \theta(1-v)^{m-j}v^{j-\frac{1}{\theta}}\frac{1}{\theta v^{1-\frac{1}{\theta}}}dv,$$

$$= \frac{1}{\theta^2}C_{j,m}\int_0^1 \left(1 + j\log v - \frac{(m-j)v\log v}{(1-v)}\right)^2 (1-v)^{m-j}v^{j-1}dv.$$

The values of $\theta^2 I_{Y_{j:m}}(\theta)$ for Beta$(\theta, 1)$ are presented in Table 2.12 for different $j$ and $m$.

Table 2.12 The values of $\theta^2 I_{Y_{j:m}}(\theta)$ for Beta$(\theta,1)$.

| $m$ | $j$ | $\theta^2 I_{Y_{j:m}}(\theta)$ | $m$ | $j$ | $\theta^2 I_{Y_{j:m}}(\theta)$ | $m$ | $j$ | $\theta^2 I_{Y_{j:m}}(\theta)$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 2.5 | 7 | 1 | 4.63144 | | 4 | 5.55788 |
| | 2 | 1.92468 | | 2 | 5.00361 | | 5 | 4.78933 |
| | 3 | 1 | | 3 | 4.57389 | | 6 | 3.91008 |
| 4 | 1 | 3.11 | | 4 | 3.8311 | | 7 | 2.96875 |
| | 2 | 2.77778 | | 5 | 2.94444 | | 8 | 1.99312 |
| | 3 | 1.96048 | | 6 | 1.98828 | | 9 | 1 |
| | 4 | 1 | | 7 | 1 | | | |
| 5 | 1 | 3.66204 | 8 | 1 | 5.06532 | 10 | 1 | 5.85612 |
| | 2 | 3.56944 | | 2 | 5.65963 | | 2 | 6.87407 |
| | 3 | 2.875 | | 3 | 5.36537 | | 3 | 6.84973 |
| | 4 | 1.97579 | | 4 | 4.70987 | | 4 | 6.37702 |
| | 5 | 1 | | 5 | 3.87982 | | 5 | 5.67343 |
| 6 | 1 | 4.16583 | | 6 | 2.95918 | | 6 | 4.83992 |
| | 2 | 4.30889 | | 7 | 1.99118 | | 7 | 3.93017 |
| | 3 | 3.745 | | 8 | 1 | | 8 | 2.97531 |
| | 4 | 2.92 | 9 | 1 | 5.4723 | | 9 | 1.99449 |
| | 5 | 1.98369 | | 2 | 6.28179 | | 10 | 1 |
| | 6 | 1 | | 3 | 6.12294 | | | |

From Table 2.12, we note that the amount of information about $\theta$, collected by $Y_{j:m}$, attains its maximum at $j = 1,\ 1 \le m \le 5$ and at $j = 2,\ 6 \le m \le 10$.

**b. Beta$(1,\theta)$**

Let $X_1, X_2, \dots, X_m$ be iid Beta$(1,\theta)$. The pdf of $Y_{j:m}$ is given by

$$f_{Y_{j:m}}(y) = \frac{m!}{(j-1)!\,(m-j)!}\theta(1-y)^{\theta-1}\left[1-(1-y)^{\theta}\right]^{j-1}(1-y)^{\theta(m-j)},$$

$$= C_{j,m}\theta(1-y)^{\theta(m+1-j)-1}\left[1-(1-y)^{\theta}\right]^{j-1} \quad ,$$

Let $f^* = logf_{Y_{j:m}}(y)$, then

$$f^* = logf_{Y_{j:m}}(y) = logC_{j,m} + log\theta + (\theta(m+1-j)-1)\,log(1-y) +$$

$$(j-1)\,log\left(1-(1-y)^{\theta}\right),$$

Taking the derivative of $f^*$ with respect to $\theta$, then

$$\frac{\partial f^*}{\partial \theta} = \frac{1}{\theta} + (1+m-j)\, log(1-y) - \frac{(j-1)(1-y)^\theta log(1-y)}{(1-(1-y)^\theta)},$$

The Fisher information about $\theta$ in $Y_{j:m}$ is

$$I_{Y_{j:m}}(\theta) = E\left(\frac{1}{\theta} + (1+m-j)\, log\left(1-Y_{j:m}\right) - \frac{(j-1)\left(1-Y_{j:m}\right)^\theta log\left(1-Y_{j:m}\right)}{(1-(1-Y_{j:m})^\theta)}\right)^2,$$

$$= C_{j,m}\int_0^1 \left(\frac{1}{\theta} + (1+m-j)\, log(1-y) - \frac{(j-1)(1-y)^\theta log(1-y)}{(1-(1-y)^\theta)}\right)^2 \times$$

$$\theta(1-y)^{\theta(1+m-j)-1}\left(1-(1-y)^\theta\right)^{j-1}\; dy.$$

Using the transformation $v = (1-y)^\theta$, we get

$$I_{Y_{j:m}}(\theta) = C_{j,m}\int_0^1 \left(\frac{1}{\theta} + (1+m-j)\frac{logv}{\theta} - \frac{(j-1)vlogv}{\theta(1-v)}\right)^2 \frac{\theta v^{(1+m-j)-\frac{1}{\theta}}}{\theta v^{1-\frac{1}{\theta}}}(1-v)^{j-1}dv,$$

$$= \frac{C_{j,m}}{\theta^2}\int_0^1 \left(1 + (1+m-j)logv - \frac{(j-1)vlogv}{(1-v)}\right)^2 v^{m-j}(1-v)^{j-1}dv.$$

The values of $\theta^2 I_{Y_{j:m}}(\theta)$ for Beta$(1,\theta)$ is the values of $\theta^2 I_{Y_{j:m}}(\theta)$ for Beta$(\theta,1)$ but in a reversed order. The reason is because if $x\sim$Beta$(\theta,1)$, then $y = 1-x \sim$Beta$(1,\theta,)$.

**c. Power function distribution**

Let $X_1, X_2, \ldots, X_m$ be iid from Power function distribution. i.e., from the pdf

$$f(x;\, \theta) = \frac{\theta}{1-\theta} x^{\frac{2\theta-1}{1-\theta}}\,; 0 < x < 1, 0 < \theta < 1.$$

The pdf of $Y_{j:m}$ is given by

$$f_{Y_{j:m}}(y;\,\theta) = C_{j,m}\frac{\theta}{1-\theta}y^{\frac{2\theta-1}{1-\theta}}y^{\frac{\theta(j-1)}{1-\theta}}\left(1-y^{\frac{\theta}{1-\theta}}\right)^{m-j},$$

$$= C_{j,m}\frac{\theta}{1-\theta}y^{\frac{\theta(j+1)-1}{1-\theta}}\left(1-y^{\frac{\theta}{1-\theta}}\right)^{m-j}.$$

Let $f^* = logf_{Y_{j:m}}(y;\,\theta)$. Then

$$f^* = logC_{j,m} + log\theta - log(1-\theta) + \left(\frac{\theta(j+1)-1}{1-\theta}\right)logy + (m-j)log\left(1-y^{\frac{\theta}{1-\theta}}\right),$$

$$= logC_{j,m} + log\theta - log(1-\theta) + \left(\frac{\theta(j+1)}{1-\theta}\right)logy - \frac{1}{(1-\theta)}logy +$$

$$(m-j)log\left(1-y^{\frac{\theta}{1-\theta}}\right).$$

Taking the first derivative with respect to $\theta$ , we get

$$\frac{\partial f^*}{\partial\theta} = \frac{1}{\theta} + \frac{1}{1-\theta} + \frac{jlogy}{(1-\theta)^2} - \frac{(m-j)y^{\frac{\theta}{1-\theta}}logy}{(1-\theta)^2(1-y^{\frac{\theta}{1-\theta}})},$$

$$= \frac{1}{\theta(1-\theta)} + \frac{jlogy}{(1-\theta)^2} - \frac{(m-j)y^{\frac{\theta}{1-\theta}}logy}{(1-y^{\frac{\theta}{1-\theta}})(1-\theta)^2}.$$

The Fisher information about $\theta$ in $Y_{j:m}$ is

$$I_{Y_{j:m}}(\theta) = E\left(\frac{1}{\theta(1-\theta)} + \frac{jlogY_{j:m}}{(1-\theta)^2} - \frac{(m-j){Y_{j:m}}^{\frac{\theta}{1-\theta}}logY_{j:m}}{(1-\theta)^2(1-{Y_{j:m}}^{\frac{\theta}{1-\theta}})}\right)^2,$$

$$I_{Y_{j:m}}(\theta) = C_{j,m}\int_0^1\left(\frac{1}{\theta(1-\theta)} + \frac{jlogy}{(1-\theta)^2} - \frac{(m-j)y^{\frac{\theta}{1-\theta}}logy}{(1-\theta)^2(1-y^{\frac{\theta}{1-\theta}})}\right)^2 \times$$

$$\frac{\theta}{1-\theta}y^{\frac{\theta(j+1)-1}{1-\theta}}\left(1-y^{\frac{\theta}{1-\theta}}\right)^{m-j}dy.$$

Using the transformation $v = y^{\frac{\theta}{1-\theta}}$, we get

$$I_{Y_{j:m}}(\theta) = C_{j,m} \int_0^1 \left( \frac{1}{\theta(1-\theta)} + \frac{j(1-\theta)logv}{\theta(1-\theta)^2} - \frac{(m-j)(1-\theta)vlogv}{\theta(1-v)(1-\theta)^2} \right)^2 \times$$

$$v^{j-1}(1-v)^{m-j}dv,$$

$$I_{Y_{j:m}}(\theta) = C_{j,m} \int_0^1 \left( \frac{1}{\theta(1-\theta)} + \frac{jlogv}{\theta(1-\theta)} - \frac{(m-j)vlogv}{\theta(1-\theta)(1-v)} \right)^2 v^{j-1}(1-v)^{m-j}dv,$$

$$= \frac{C_{j,m}}{\theta^2(1-\theta)^2} \int_0^1 \left( 1 + jlogv - \frac{(m-j)vlogv}{(1-v)} \right)^2 v^{j-1}(1-v)^{m-j}dv.$$

The values of $\theta^2(1-\theta)^2 I_{Y_{j:m}}(\theta)$ for power function are presented in Table 2.13 for different $j$ and $m$.

Table 2.13 The values of $\theta^2(1-\theta)^2 I_{Y_{j:m}}(\theta)$ for power function.

| m | j | $\theta^2(1-\theta)^2 I_{Y_{j:m}}(\theta)$ | m | j | $\theta^2(1-\theta)^2 I_{Y_{j:m}}(\theta)$ | m | j | $\theta^2(1-\theta)^2 I_{Y_{j:m}}(\theta)$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 2.5 | 7 | 1 | 4.63144 | 9 | 4 | 5.55788 |
| | 2 | 1.92468 | | 2 | 5.00361 | | 5 | 4.78933 |
| | 3 | 1 | | 3 | 4.57389 | | 6 | 3.91008 |
| 4 | 1 | 3.11111 | | 4 | 3.83111 | | 7 | 2.96875 |
| | 2 | 2.77778 | | 5 | 2.94444 | | 8 | 1.99312 |
| | 3 | 1.96048 | | 6 | 1.98828 | | 9 | 1 |
| | 4 | 1 | | 7 | 1 | | | |
| 5 | 1 | 3.66204 | 8 | 1 | 5.06532 | 10 | 1 | 5.85612 |
| | 2 | 3.56944 | | 2 | 5.65963 | | 2 | 6.87407 |
| | 3 | 2.875 | | 3 | 5.36537 | | 3 | 6.84973 |
| | 4 | 1.97579 | | 4 | 4.70987 | | 4 | 6.37702 |
| | 5 | 1 | | 5 | 3.87982 | | 5 | 5.67343 |
| 6 | 1 | 4.16583 | | 6 | 2.95918 | | 6 | 4.83992 |
| | 2 | 4.30889 | | 7 | 1.99118 | | 7 | 3.93017 |
| | 3 | 3.745 | | 8 | 1 | | 8 | 2.97531 |
| | 4 | 2.92 | 9 | 1 | 5.4723 | | 9 | 1.99449 |
| | 5 | 1.98369 | | 2 | 6.28179 | | 10 | 1 |
| | 6 | 1 | | 3 | 6.12294 | | | |

From Table 2.13, we note that the amount of information about $\theta$, collected by $Y_{j:m}$, attains its maximum at $j = 1,\ 1 \le m \le 5$ and at $j = 2,\ 6 \le m \le 10$.

**d. Pareto distribution**

Let $X_1, X_2, \dots, X_m$ be iid Pareto$(\alpha, \beta)$ with known $\alpha$. The pdf of $Y_{j:m}$ is given by

$$f_{Y_{j:m}}(y) = \frac{C_{j,m}\beta\alpha^{\beta}}{y^{\beta+1}}\left(1-\left(\frac{\alpha}{y}\right)^{\beta}\right)^{j-1}\left(\frac{\alpha}{y}\right)^{\beta(m-j)}.$$

The log likelihood is

$$f^* = logf_{Y_{j:m}}(y) = logC_{j,m} + log\beta + \beta log\alpha - (\beta+1)logy +$$

$$\beta(m-j)log\left(\frac{\alpha}{y}\right) + (j-1)log\left(1-\left(\frac{\alpha}{y}\right)^{\beta}\right).$$

Taking the derivative of $f^*$ with respect to $\beta$, we get

$$\frac{\partial f^*}{\partial \beta} = log\alpha + \frac{1}{\beta} - logy + (m-j)log\left(\frac{\alpha}{y}\right) - \frac{(j-1)\left(\frac{\alpha}{y}\right)^{\beta} log\left(\frac{\alpha}{y}\right)}{\left(1-\left(\frac{\alpha}{y}\right)^{\beta}\right)}.$$

The Fisher information about $\beta$ in $Y_{j:m}$ is

$$I_{Y_{j:m}}(\beta) = E\left(\frac{\partial logf_{Y_{j:m}}(Y_{j:m})}{\partial \beta}\right)^2.$$

Then we note that, the Fisher information for Pareto$(1, \beta)$ is similar to the Fisher information for Beta$(1, \theta)$.

**e. Weibull $(\gamma, \beta)$**

Case 1. $\gamma$ is known

Let $X_1, X_2, \dots, X_m$ be iid Weibull $(\gamma, \beta)$, $\gamma > 0$, $\beta > 0$. The pdf of $Y_{j:m}$ is given by

$$f_{Y_{j:m}}(y) = \frac{C_{j,m}\gamma}{\beta}\left(1 - e^{-\left(\frac{y}{\beta}\right)^{\gamma}}\right)^{j-1} e^{-(m-j)\left(\frac{y}{\beta}\right)^{\gamma}}\left(\frac{y}{\beta}\right)^{\gamma-1} e^{-\left(\frac{y}{\beta}\right)^{\gamma}},$$

$$= \frac{C_{j,m}\gamma}{\beta}\left(1-e^{-\left(\frac{y}{\beta}\right)^{\gamma}}\right)^{j-1} e^{-(m-j+1)\left(\frac{y}{\beta}\right)^{\gamma}} \left(\frac{y}{\beta}\right)^{\gamma-1}.$$

The log likelihood is

$$f^{*} = logC_{j,m}\gamma - log\beta + c_1 log\left(1-e^{-\left(\frac{y}{\beta}\right)^{\gamma}}\right) - c_2\left(\frac{y}{\beta}\right)^{\gamma} + (\gamma-1)log\left(\frac{y}{\beta}\right),$$

$$= logC_{j,m}\gamma - \gamma log\beta + c_1 log\left(1-e^{-\left(\frac{y}{\beta}\right)^{\gamma}}\right) - c_2\left(\frac{y}{\beta}\right)^{\gamma} + (\gamma-1)logy,$$

where $f^{*} = logf_{Y_{j:m}}(y)$, $c_1 = (j-1)$ and $c_2 = (m-j+1)$.

Taking the derivative of $f^{*}$ with respect to $\beta$, we get

$$\frac{\partial f^{*}}{\partial \beta} = -\frac{\gamma}{\beta} - \frac{c_1\gamma y^{\gamma}\beta^{-(\gamma+1)}e^{-\left(\frac{y}{\beta}\right)^{\gamma}}}{1-e^{-\left(\frac{y}{\beta}\right)^{\gamma}}} + c_2\gamma y^{\gamma}\beta^{-(\gamma+1)}.$$

The Fisher information about $\beta$ in $Y_{j:m}$ is

$$I_{Y_{j:m}}(\beta) = E\left(-\frac{\gamma}{\beta} - \frac{c_1\gamma Y_{j:m}^{\gamma}\beta^{-(\gamma+1)}e^{-\left(\frac{Y_{j:m}}{\beta}\right)^{\gamma}}}{1-e^{-\left(\frac{Y_{j:m}}{\beta}\right)^{\gamma}}} + c_2\gamma Y_{j:m}^{\gamma}\beta^{-(\gamma+1)}\right)^{2},$$

$$= C_{j,m}\int_{0}^{\infty}\left(-\frac{\gamma}{\beta} - \frac{c_1\gamma y^{\gamma}\beta^{-(\gamma+1)}e^{-\left(\frac{y}{\beta}\right)^{\gamma}}}{1-e^{-\left(\frac{y}{\beta}\right)^{\gamma}}} + c_2\gamma y^{\gamma}\beta^{-(\gamma+1)}\right)^{2} \times$$

$$\frac{\gamma}{\beta}\left(1-e^{-\left(\frac{y}{\beta}\right)^{\gamma}}\right)^{c_1} e^{-c_2\left(\frac{y}{\beta}\right)^{\gamma}}\left(\frac{y}{\beta}\right)^{\gamma-1} dy.$$

Using the transformation $v = \left(\frac{y}{\beta}\right)^{\gamma}$ we get

$$I_{Y_{j:m}}(\beta) = \frac{C_{j,m}}{\beta^2} \int_0^{\infty} \left( -\gamma - \frac{c_1 \gamma v e^{-v}}{1 - e^{-v}} + c_2 \gamma v \right)^2 (1 - e^{-v})^{c_1} e^{-c_2 v} dv.$$

The values of $\beta^2 I_{Y_{j:m}}(\beta)$ for Weibull $(1, \beta)$ are presented in Table 2.14 for different $j$ and $m$.

Table 2.14 The values of $\beta^2 I_{Y_{j:m}}(\beta)$ for Weibull$(1, \beta)$.

| $m$ | $j$ | $\beta^2 I_{Y_{j:m}}(\beta)$ | $m$ | $j$ | $\beta^2 I_{Y_{j:m}}(\beta)$ | $m$ | $j$ | $\beta^2 I_{Y_{j:m}}(\beta)$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 1 | 7 | 1 | 1 | 9 | 4 | 3.91008 |
| | 2 | 1.92468 | | 2 | 1.98828 | | 5 | 4.78933 |
| | 3 | 2.5 | | 3 | 2.94444 | | 6 | 5.55788 |
| 4 | 1 | 1 | | 4 | 3.83111 | | 7 | 6.12294 |
| | 2 | 1.96048 | | 5 | 4.57389 | | 8 | 6.28179 |
| | 3 | 2.77778 | | 6 | 5.00361 | | 9 | 5.4723 |
| | 4 | 3.111111 | | 7 | 4.63144 | | | |
| 5 | 1 | 1 | 8 | 1 | 1 | 10 | 1 | 1 |
| | 2 | 1.97579 | | 2 | 1.99118 | | 2 | 1.99449 |
| | 3 | 2.875 | | 3 | 2.95918 | | 3 | 2.97531 |
| | 4 | 3.56944 | | 4 | 3.87982 | | 4 | 3.93017 |
| | 5 | 3.66204 | | 5 | 4.70987 | | 5 | 4.83992 |
| 6 | 1 | 1 | | 6 | 5.36537 | | 6 | 5.67343 |
| | 2 | 1.98369 | | 7 | 5.65963 | | 7 | 6.37702 |
| | 3 | 2.92 | | 8 | 5.06532 | | 8 | 6.84973 |
| | 4 | 3.745 | 9 | 1 | 1 | | 9 | 6.87407 |
| | 5 | 4.30889 | | 2 | 1.99312 | | 10 | 5.85612 |
| | 6 | 4.16583 | | 3 | 2.96875 | | | |

From Table 2.14, we note that the amount of information about $\beta$, collected by $Y_{j:m}$, attains its maximum at $j = m, 1 \le m \le 5, j = m - 1, \ 6 \le m \le 10$.

Case 2. $\beta$ is known

In this case we derive $f^* = log f_{Y_{j:m}}(y)$ with respect to $\gamma$, then we get

$$\frac{\partial f^*}{\partial \gamma} = \frac{1}{\gamma} + \frac{c_1 \left(\frac{y}{\beta}\right)^{\gamma} e^{-\left(\frac{y}{\beta}\right)^{\gamma}} log\left(\frac{y}{\beta}\right)}{1 - e^{-\left(\frac{y}{\beta}\right)^{\gamma}}} - c_2 \left(\frac{y}{\beta}\right)^{\gamma} log\left(\frac{y}{\beta}\right) + logy - log\beta.$$

The Fisher information about $\gamma$ in $Y_{j:m}$ is

$$I_{Y_{j:m}}(\gamma) = E\left(\frac{\partial f^*}{\partial \gamma}\right)^2,$$

$$= \int_0^\infty \left(\frac{\partial f^*}{\partial \gamma}\right)^2 \frac{C_{j,m}\gamma}{\beta}\left(1 - e^{-\left(\frac{y}{\beta}\right)^\gamma}\right)^{c_1} e^{-c_2\left(\frac{y}{\beta}\right)^\gamma} \left(\frac{y}{\beta}\right)^{\gamma-1} dy.$$

Using the transformation $v = \left(\frac{y}{\beta}\right)^\gamma$ we get

$$I_{Y_{j:m}}(\gamma) = \frac{C_{j,m}}{\gamma^2} \int_0^\infty \left(1 + \frac{c_1 v e^{-v} logv}{(1-e^{-v})} - c_2 v logv + logv\right)^2 e^{-c_2 v}(1-e^{-v})^{c_1} dv.$$

The values of $\gamma^2 I_{Y_{j:m}}(\gamma)$ for Weibull( $\gamma$, 1) are presented in Table 2.15 for different $j$ and $m$.

Table 2.15 The values of $\gamma^2 I_{Y_{j:m}}(\gamma)$ for Weibull($\gamma$, 1).

| $m$ | $j$ | $\gamma^2 I_{Y_{j:n}}(\gamma)$ | $m$ | $j$ | $\gamma^2 I_{Y_{j:n}}(\gamma)$ | $m$ | $j$ | $\gamma^2 I_{Y_{j:n}}(\gamma)$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 2.10168 | 7 | 1 | 3.96485 | 9 | 4 | 2.83332 |
| | 2 | 1.79061 | | 2 | 3.56858 | | 5 | 2.11082 |
| | 3 | 3.09013 | | 3 | 2.65766 | | 6 | 1.93737 |
| 4 | 1 | 2.57329 | | 4 | 1.99114 | | 7 | 2.58226 |
| | 2 | 1.98807 | | 5 | 2.02365 | | 8 | 4.33884 |
| | 3 | 1.97168 | | 6 | 3.19102 | | 9 | 7.3884 |
| | 4 | 3.78444 | | 7 | 5.94852 | | | |
| 5 | 1 | 3.05308 | 8 | 1 | 4.38945 | 10 | 1 | 5.17858 |
| | 2 | 2.4417 | | 2 | 4.15449 | | 2 | 5.30227 |
| | 3 | 1.8793 | | 3 | 3.22312 | | 3 | 4.45681 |
| | 4 | 2.28305 | | 4 | 2.34301 | | 4 | 3.40251 |
| | 5 | 4.4992 | | 5 | 1.91146 | | 5 | 2.49433 |
| 6 | 1 | 3.51903 | | 6 | 2.26003 | | 6 | 1.97389 |
| | 2 | 2.98885 | | 7 | 3.7428 | | 7 | 2.05762 |
| | 3 | 2.17916 | | 8 | 6.67128 | | 8 | 2.97499 |
| | 4 | 1.89101 | 9 | 1 | 4.79357 | | 9 | 4.9684 |
| | 5 | 2.69795 | | 2 | 4.73434 | | 10 | 8.09807 |
| | 6 | 5.22297 | | 3 | 3.83075 | | | |

From Table 2.15, we note that the amount of information about $\gamma$, collected by $Y_{j:m}$, attains its maximum at $j = m$ for all $m$.

## f. LL($\alpha, \beta$)

Case 1. $\beta$ is known

Let $X_1, X_2, \ldots, X_m$ be iid LL $(\alpha, \beta)$, $\alpha > 0$, $\beta > 0$. The pdf of $Y_{j:m}$ is given by

$$f_{Y_{j:m}}(y) = \frac{C_{j,m}\beta(y/\alpha)^{\beta-1}}{\alpha\left(1+\left(\frac{y}{\alpha}\right)^{\beta}\right)^{2}}\left(1-\frac{1}{1+\left(\frac{\alpha}{y}\right)^{\beta}}\right)^{m-j}\left(\frac{1}{1+\left(\frac{\alpha}{y}\right)^{\beta}}\right)^{j-1}.$$

The log likelihood is

$$f^* = logC_{j,m} + log\beta - log\alpha + (\beta-1)log\left(\frac{y}{\alpha}\right) - 2log\left(1+\left(\frac{y}{\alpha}\right)^{\beta}\right) +$$

$$c_2 log\left(1-\frac{1}{1+\left(\frac{\alpha}{y}\right)^{\beta}}\right) + c_1 log\left(\frac{1}{1+\left(\frac{\alpha}{y}\right)^{\beta}}\right).$$

where $f^* = logf_{Y_{j:m}}(y)$, $c_1 = j-1$ and $c_2 = m-j$.

Taking the derivative of $f^*$ with respect to $\alpha$, we get

$$\frac{\partial f^*}{\partial \alpha} = -\frac{1}{\alpha} - \frac{(\beta-1)}{\alpha} + \frac{2\beta\alpha^{-(1+\beta)}y^{\beta}}{1+\left(\frac{y}{\alpha}\right)^{\beta}} - \frac{c_1\beta\alpha^{\beta-1}}{y^{\beta}\left(1+\left(\frac{\alpha}{y}\right)^{\beta}\right)} + \frac{c_2\beta\alpha^{\beta-1}\left(1+\left(\frac{\alpha}{y}\right)^{\beta}\right)^{-2}}{y^{\beta}\left(1-\frac{1}{1+\left(\frac{\alpha}{y}\right)^{\beta}}\right)}.$$

The Fisher information about $\alpha$ in $Y_{j:m}$ is

$$I_{Y_{j:m}}(\alpha) = E\left(\frac{\partial logf_{Y_{j:m}}(y)}{\partial \alpha}\right)^{2},$$

$$= C_{j,m}\int_0^\infty \left(\frac{\partial log f_{Y_{j:m}}(y)}{\partial\alpha}\right)^2 \frac{\beta(y/\alpha)^{\beta-1}}{\alpha\left(1+\left(\frac{y}{\alpha}\right)^\beta\right)^2}\left(1-\frac{1}{1+\left(\frac{\alpha}{y}\right)^\beta}\right)^{m-j}\left(\frac{1}{1+\left(\frac{\alpha}{y}\right)^\beta}\right)^{j-1} dy.$$

Using the transformation $v = \left(\frac{y}{\alpha}\right)^\beta$., we get

$$I_{Y_{j:m}}(\alpha) = \frac{C_{j,m}}{\alpha^2}\int_0^\infty \left(-\beta+\frac{2\beta v}{1+v}-\frac{c_1\beta}{1+v}+\frac{c_2\beta v^{-1}(1+v^{-1})^{-2}}{\left(1-\frac{1}{1+v^{-1}}\right)}\right)^2 \times$$

$$\frac{1}{(1+v)^2}\left(1-\frac{1}{1+v^{-1}}\right)^{m-j}\left(\frac{1}{1+v^{-1}}\right)^{j-1} dv.$$

Calculations show that $\alpha^2 I_{Y_{j:m}}(\alpha)$, for LL($\alpha$,1), is the case as for logistic distribution obtained in Table 2.2.

Case 2. $\alpha$ is known

In this case we derive $f^* = log f_{Y_{j:m}}(y)$ with respect to $\beta$.

$$\frac{\partial f^*}{\partial\beta} = \frac{1}{\beta} + log\left(\frac{y}{\alpha}\right) - \frac{2\left(\frac{y}{\alpha}\right)^\beta log\left(\frac{y}{\alpha}\right)}{1+\left(\frac{y}{\alpha}\right)^\beta} - \frac{c_1\left(\frac{\alpha}{y}\right)^\beta log\left(\frac{\alpha}{y}\right)}{\left(\frac{1}{1+\left(\frac{\alpha}{y}\right)^\beta}\right)\left(1+\left(\frac{\alpha}{y}\right)^\beta\right)^2} +$$

$$\frac{c_2\left(\frac{\alpha}{y}\right)^\beta log\left(\frac{\alpha}{y}\right)}{\left(1-\frac{1}{1+\left(\frac{\alpha}{y}\right)^\beta}\right)\left(1+\left(\frac{\alpha}{y}\right)^\beta\right)^2}.$$

The Fisher information about $\beta$ in $Y_{j:m}$ is

$$I_{Y_{j:m}}(\beta) = E\left(\frac{\partial log f_{Y_{j:m}}(y)}{\partial \beta}\right)^2,$$

$$= C_{j,m}\int_0^\infty \left(\frac{\partial log f_{Y_{j:m}}(y)}{\partial \beta}\right)^2 \frac{\beta(y/\alpha)^{\beta-1}}{\alpha\left(1+\left(\frac{y}{\alpha}\right)^\beta\right)^2}\left(1-\frac{1}{1+\left(\frac{\alpha}{y}\right)^\beta}\right)^{m-j}\left(\frac{1}{1+\left(\frac{\alpha}{y}\right)^\beta}\right)^{j-1} dy.$$

Using the transformation $v = \left(\frac{y}{\alpha}\right)^\beta$ , we get

$$\beta^2 I_{Y_{j:m}}(\beta) = \int_0^\infty \left(1 + log v\left(1 - \frac{2v}{1+v} + \frac{c_1}{(1+v)} - \frac{c_2(1+v^{-1})^{-2}}{\left(v - \frac{v}{1+v^{-1}}\right)}\right)\right)^2 \times$$

$$\frac{C_{j,m}}{(1+v)^2}\left(1 - \frac{1}{1+v^{-1}}\right)^{m-j}\left(\frac{1}{1+v^{-1}}\right)^{j-1} dv.$$

Calculations show that $\beta^2 I_{Y_{j:m}}(\beta)$, for LL$(1,\beta)$, is the case as for logistic distribution obtained in Table 2.6.

## 2.6 Best Sampling Schemes

In this section, we characterize the best sampling schemes for the family of probability distribution considered in chapter two.

**a. Exp($\theta$)**

Let $X_1, X_2, \ldots, X_m$ be a random sample of size $m$ from exp $(\theta)$. The pdf of $Y_{j:m}$ , the $j^{th}$ order statistic is :

$$g_j(y;\theta) = \frac{C_{j,m}}{\theta}\left(1 - e^{-\frac{y}{\theta}}\right)^{j-1} e^{-(m-j+1)\frac{y}{\theta}}; \qquad y > 0,$$

where $C_{j,m} = \frac{m!}{(j-1)!(m-j)!}$.

According to Table 2.4, the amount of information about $\theta$ collected by $Y_{j:m}$ attains its maximum, as a function of $j$, at $j=m$ if $1 \le m \le 5$ and at $j=m-1$ if $6 \le m \le 10$.

So for $m \le 5$, the best sampling scheme is the one which chooses the maximum of each set for actual quantification. For $6 \le m \le 10$, chooses $Y_{m-1:m}$ for actual quantification. As $m \to \infty$, we expect to choose $Y_{1:m}$ which is $\exp\left(\frac{\theta}{m}\right)$, i.e, we choose a scheme which carries the same amount of information as a SRS of size $m$.

Table 2.16 contains the best ranked set sampling plan and the corresponding amount of information that it collects about $\theta$.

Table 2.16 The values of $I_{Y_{j:m}}(\theta)$ for Exp($\theta$) in the best RSS plan.

| $m$ | RSS plan | Fisher Information in plan |
|---|---|---|
| 3 | (3,3,3) | 7.5 |
| 4 | (4,4,4,4) | 12.4444 |
| 5 | (5,5,5,5,5) | 18.3102 |
| 6 | (5,5,5,5,5,5) | 25.8533 |
| 7 | (6,6,6,6,6,6,6) | 35.0253 |
| 8 | (7,7,7,7,7,7,7,7) | 45.277 |
| 9 | (8,8,8,8,8,8,8,8,8) | 56.5361 |
| 10 | (9,9,9,9,9,9,9,9,9,9) | 68.7407 |

Similarly, the Tables 2.16- 2.31 contain the best ranked set sampling schemes as well as the a mount of information that it collects about the parameter of interest for the distributions which we have studied.

## b. $N(\theta, 1)$

Table 2.17 The values of $I_{Y_{j:m}}(\theta)$ for N(θ, 1) in the best RSS plan.

| *m* | RSS plan | Fisher Information in plan |
|---|---|---|
| 3 | (2,2,2) | 6.68772 |
| 4 | (2,2,2,2) or (3,3,3,3) | 11.12228 |
| 5 | (3,3,3,3,3) | 17.4346 |
| 6 | (3,3,3,3,3,3) or (4,4,4,4,4,4) | 24.39132 |
| 7 | (4,4,4,4,4,4,4) | 33.26652 |
| 8 | (4,4,4,4,4,4,4,4) or (5,5,5,5,5,5,5,5) | 42.758 |
| 9 | (5,5,5,5,5,5,5,5,5) | 54.18837 |
| 10 | (5,5,5,5,5,5,5,5,5,5) or (6,6,6,6,6,6,6,6,6,6) | 66.2196 |

The amount of information about $\theta$ for LN$(\theta, 1)$ is the same of the amount of information about $\theta$ for N$(1, \theta)$.

**c. N$(0, \sigma^2)$**

Table 2.18 The values of $I_{Y_{j:m}}(\sigma)$ for N$(0, \sigma^2)$in the best RSS plan.

| *m* | RSS plan | Fisher Information in plan |
|---|---|---|
| 3 | (1,1,1) or (3,3,3) | 8.48097 |
| 4 | (1,1,1,1) or (4,4,4,4) | 13.88084 |
| 5 | (1,1,1,1,1) or (5,5,5,5,5) | 20.6945 |
| 6 | (1,1,1,1,1,1) or (6,6,6,6,6,6) | 28.85538 |
| 7 | (1,1,1,1,1,1,1) or (7,7,7,7,7,7,7) | 38.29539 |
| 8 | (1,1,1,1,1,1,1,1) or (8,8,8,8,8,8,8,8) | 48.95032 |
| 9 | (1,1,1,1,1,1,1,1,1) or (9,9,9,9,9,9,9,9,9) | 60.76134 |
| 10 | (1,1,1,1,1,1,1,1,1,1) or (10,10,10,10,10,10,10,10,10,10) | 73.6749 |

The amount of information about $\sigma^2$ for LN$(0, \sigma^2)$ is the same of the amount of information about $\sigma^2$ for N$(0, \sigma^2)$.

**d. $N(\theta, \theta^2)$**

Table 2.19 The values of $I_{Y_{j:m}}(\theta)$ for $N(\theta, \theta^2)$ in the best RSS plan.

| $m$ | RSS plan | Fisher Information in plan |
|---|---|---|
| 3 | (3,3,3) | 21.576 |
| 4 | (4,4,4,4) | 36.67132 |
| 5 | (5,5,5,5,5) | 55.09 |
| 6 | (6,6,6,6,6,6) | 76.521 |
| 7 | (7,7,7,7,7,7,7) | 100.7132 |
| 8 | (8,8,8,8,8,8,8,8) | 127.4584 |
| 9 | (9,9,9,9,9,9,9,9,9) | 156.582 |
| 10 | (9,9,9,9,9,9,9,9,9,9) | 192.91 |

**e. $Beta(\theta, 1)$**

Table 2.20 The values of $I_{Y_{j:m}}(\theta)$ for $Beta(\theta, 1)$in the best RSS plan .

| $m$ | RSS plan | Fisher Information in plan |
|---|---|---|
| 3 | (1,1,1) | 7.5 |
| 4 | (1,1,1,1) | 12.44 |
| 5 | (1,1,1,1,1) | 18.3102 |
| 6 | (2,2,2,2,2,2) | 25.85334 |
| 7 | (2,2,2,2,2,2,2) | 32.02527 |
| 8 | (2,2,2,2,2,2,2,2) | 45.27704 |
| 9 | (2,2,2,2,2,2,2,2,2) | 56.53611 |
| 10 | (2,2,2,2,2,2,2,2,2,2) | 68.7407 |

Table 2.21 The values of $I_{Y_{j:m}}(\theta)$ for $Beta(1, \theta)$in the best RSS plan.

| Number of $m$ | RSS plan | Fisher Information in plan |
|---|---|---|
| 3 | (3,3,3) | 7.5 |
| 4 | (4,4,4,4) | 12.44 |
| 5 | (5,5,5,5,5) | 18.3102 |
| 6 | (5,5,5,5,5,5) | 25.85334 |
| 7 | (6,6,6,6,6,6,6) | 32.02527 |
| 8 | (7,7,7,7,7,7,7,7) | 45.27704 |
| 9 | (8,8,8,8,8,8,8,8,8) | 56.53611 |
| 10 | (9,9,9,9,9,9,9,9,9,9) | 68.7407 |

The amount of information about $\theta$ for Beta$(1,\theta)$ is the same of the amount of information about $\beta$ for Pareto$(1,\beta)$.

**g. Power distribution**

Table 2.22 The values of $I_{Y_{j:m}}(\theta)$ for Power distribution in the best RSS plan.

| $m$ | RSS plan | Fisher Information in plan |
|---|---|---|
| 3 | (1,1,1) | 7.5 |
| 4 | (1,1,1,1) | 12.444444 |
| 5 | (1,1,1,1,1) | 18.3102 |
| 6 | (2,2,2,2,2,2) | 25.85334 |
| 7 | (2,2,2,2,2,2,2) | 35.02527 |
| 8 | (2,2,2,2,2,2,2,2) | 45.27704 |
| 9 | (2,2,2,2,2,2,2,2,2) | 56.53611 |
| 10 | (2,2,2,2,2,2,2,2,2,2) | 68.7407 |

**h. Cauchy distribution**

Table 2.23 The values of $I_{Y_{j:m}}(\theta)$ for Cauchy$(\theta,1)$ in the best RSS plan.

| $m$ | RSS plan | Fisher Information in plan |
|---|---|---|
| 3 | (2,2,2) | 3.27192 |
| 4 | (2,2,2,2) or (3,3,3,3) | 5.27444 |
| 5 | (3,3,3,3,3) | 8.7276 |
| 6 | (3,3,3,3,3,3) or (4,4,4,4,4,4) | 12.04608 |
| 7 | (4,4,4,4,4,4,4) | 17.08294 |
| 8 | (4,4,4,4,4,4,4,4) or (5,5,5,5,5,5,5,5 ) | 21.80192 |
| 9 | (5,5,5,5,5,5,5,5,5) | 28.45476 |
| 10 | (5,5,5,5,5,5,5,5,5,5) or (6,6,6,6,6,6,6,6,6,6) | 34.6274 |

Table 2.24 The values of $I_{Y_{j:m}}(\sigma)$ for Cauchy $(0,\sigma)$ in the best RSS plan.

| $m$ | RSS plan | Fisher Information in plan |
|---|---|---|
| 3 | (2,2,2) | 2.348565 |
| 4 | (2,2,2,2) or (3,3,3,3) | 3.435384 |
| 5 | (3,3,3,3,3) | 4.86452 |
| 6 | (3,3,3,3,3,3) or (4,4,4,4,4,4,) | 6.24624 |
| 7 | (4,4,4,4,4,4,4) | 7.76965 |
| 8 | (3,3,3,3,3,3,3,3) or (6,6,6,6,6,6,6,6) | 9.51288 |
| 9 | (3,3,3,3,3,3,3,3,3) or (7,7,7,7,7,7,7,7,7) | 11.54583 |
| 10 | (3,3,3,3,3,3,3,3,3,3) or (8,8,8,8,8,8,8,8,8,8) | 13.8576 |

Table 2.25 The values of $I_{Y_{j:m}}(\theta)$ for Cauchy distribution with $c$=1 in the best RSS plan.

| $m$ | RSS plan | Fisher Information in plan |
|---|---|---|
| 3 | (2,2,2) | 5.61399 |
| 4 | (3,3,3,3) | 9.60908 |
| 5 | (4,4,4,4,4) | 14.09585 |
| 6 | (4,4,4,4,4,4) | 20.35224 |
| 7 | (5,5,5,5,5,5,5) | 27.80358 |
| 8 | (6,6,6,6,6,6,6,6) | 35.65696 |
| 9 | (6,6,6,6,6,6,6,6,6) | 45.35784 |
| 10 | (7,7,7,7,7,7,7,7,7,7) | 56.5434 |

**g. Weibull distribution**

Table 2.26The values of $I_{Y_{j:m}}(\beta)$ for Weibull$(1,\beta)$ in the best RSS plan.

| $m$ | RSS plan | Fisher Information in plan |
|---|---|---|
| 3 | (3,3,3) | 7.5 |
| 4 | (4,4,4,4) | 12.4444 |
| 5 | (5,5,5,5,5) | 18.3102 |
| 6 | (5,5,5,5,5,5) | 25.85334 |
| 7 | (6,6,6,6,6,6,6) | 35.02527 |
| 8 | (7,7,7,7,7,7,7,7) | 45.27704 |
| 9 | (8,8,8,8,8,8,8,8,8) | 56.53611 |
| 10 | (9,9,9,9,9,9,9,9,9,9) | 68.7407 |

Table 2.27 The values of $I_{Y_{j:m}}(\gamma)$ for Weibull$(\gamma,1)$ in the best RSS plan.

| $m$ | RSS plan | Fisher Information in plan |
|---|---|---|
| 3 | (3,3,3) | 9.27039 |
| 4 | (4,4,4,4) | 15.13776 |
| 5 | (5,5,5,5,5) | 22.496 |
| 6 | (6,6,6,6,6,6) | 31.33782 |
| 7 | (7,7,7,7,7,7,7) | 41.63964 |
| 8 | (8,8,8,8,8,8,8,8) | 53.37024 |
| 9 | (9,9,9,9,9,9,9,9,9) | 66.4956 |
| 10 | (10,10,10,10,10,10,10,10,10,10) | 80.9807 |

### i. Log Normal distribution

Table 2.28 The values of $I_{Y_{j:m}}(\theta)$ for LN$(\theta,\sigma^2)$ with $c$=1 in the best RSS plan.

| $m$ | RSS plan | Fisher Information in plan |
|---|---|---|
| 3 | (2,2,2) | 9.64836 |
| 4 | (2,2,2,2) or (3,3,3,3) | 16.04604 |
| 5 | (3,3,3,3,3) | 25.1528 |
| 6 | (3,3,3,3,3,3) or (4,4,4,4,4,4) | 35.18922 |
| 7 | (4,4,4,4,4,4,4) | 47.99347 |
| 8 | (4,4,4,4,4,4,4,4) or (5,5,5,5,5,5,5,5 ) | 61.68672 |
| 9 | (5,5,5,5,5,5,5,5,5) | 78.17751 |
| 10 | (5,5,5,5,5,5,5,5,5,5) or (6,6,6,6,6,6,6,6,6,6) | 95.5347 |

### j. Logistic distribution

Table 2.29 The values of $I_{Y_{j:m}}(\theta)$ for logistic$(\theta, 1)$ in the best RSS plan.

| $m$ | RSS plan | Fisher Information in plan |
|---|---|---|
| 3 | (2,2,2) | 2.4 |
| 4 | (2,2,2,2) or (3,3,3,3) | 4 |
| 5 | (3,3,3,3,3) | 6.42855 |
| 6 | (3,3,3,3,3,3) or (4,4,4,4,4,4) | 9 |
| 7 | (4,4,4,4,4,4,4) | 12.44446 |
| 8 | (4,4,4,4,4,4,4,4) or (5,5,5,5,5,5,5,5) | 16 |
| 9 | (5,5,5,5,5,5,5,5,5) | 20.45457 |
| 10 | (5,5,5,5,5,5,5,5,5,5) or(6,6,6,6,6,6,6,6,6,6) | 25 |

The amount of information about $\theta$ for Logistic$(\theta, 1)$ is the same of the amount of information about $\alpha$ for LL$(\alpha, 1)$.

Table 2.30 The values of $I_{Y_{j:m}}(\sigma)$for Logistic$(0, \sigma)$ in the best RSS plan.

| $m$ | RSS plan | Fisher Information in plan |
|---|---|---|
| 3 | (1,1,1) or (3,3,3) | 5.92176 |
| 4 | (1,1,1,1) or (4,4,4,4) | 9.43964 |
| 5 | (1,1,1,1,1) or (5,5,5,5,5) | 13.83285 |
| 6 | (1,1,1,1,1,1) or (6,6,6,6,6,6) | 19.05438 |
| 7 | (1,1,1,1,1,1,1) or (7,7,7,7,7,7,7) | 25.04978 |
| 8 | (1,1,1,1,1,1,1,1) or (8,8,8,8,8,8,8,8) | 31.76624 |
| 9 | (1,1,1,1,1,1,1,1,1) or (9,9,9,9,9,9,9,9,9) | 39.15576 |
| 10 | (1,1,1,1,1,1,1,1,1,1) or (10,10,10,10,10,10,10,10,10,10) | 47.1752 |

The amount of information about $\sigma$ for logistic$(0,\sigma)$ is the same of the amount of information about $\beta$ for LL$(1,\beta)$. Finally,

Table 2.31 The values of $I_{Y_{j:m}}(\theta)$ for logistic$(\theta,\sigma)$ with $c$=1in the best RSS plan.

| $m$ | RSS plan | Fisher Information in plan |
|---|---|---|
| 3 | (3,3,3) | 17.2849 |
| 4 | (4,4,4,4) | 28.6923 |
| 5 | (5,5,5,5,5) | 42.2088 |
| 6 | (5,5,5,5,5,5) | 60.9102 |
| 7 | (6,6,6,6,6,6,6) | 83.4126 |
| 8 | (7,7,7,7,7,7,7,7) | 108.539 |
| 9 | (8,8,8,8,8,8,8,8,8) | 136.026 |
| 10 | (9,9,9,9,9,9,9,9,9,9) | 165.656 |

## 2.7 Concluding remarks

Based on Tables 2.1- 2.21 we may write the following conclusions.

1. For a symmetric distribution about the parameter $\theta$, the best sampling scheme is the one which quantifies the median of each set provided that the scale parameter is known.
2. For a scale distribution with scale parameter $\sigma$, the best sampling scheme is that one which quantifies an extreme order statistic or an order statistic near to an extreme order statistic, depending on direction of the population skewness and provided that the location parameter is known.
3. For a scale – location distribution with known c. v., the best sampling scheme is the one which quantifies an extreme order statistic or an order statistic near the extreme order statistics except for log normal with known cv where the best sampling scheme is the one which quantifies the median.

# Chapter Three

## Estimators Based on Best Sampling Schemes

In this chapter, we use the best RSS schemes that we have obtained in the chapter two to obtain estimators to the parameters of interest for three parametric families: the location family, the scale family and the location- scale family with known coefficient of variation. Also, some distributions outside these families are considered. Moreover, we compare these estimators to their counterparts under the McIntyre RSS scheme via their efficiencies.

### 3.1 Estimators for location distribution

**a. $N(\theta, 1)$**

Let $(j_0, \dots, j_0)$ be the best ranked set sampling scheme for $N(\theta, 1)$ obtained in chapter two. The pdf $Y_{j_0:m}$ is

$$g_{j_0}(y;\theta) = C_{j_0,m}\, \Phi^{j_0-1}(y-\theta)\Phi^{m-j_0}(\theta-y)\ \phi(y-\theta),$$

and the mean of $Y_{j_0:m}$ is

$$E(Y_{j_0:m}) = \int_{-\infty}^{\infty} y\, C_{j_0,m}\, \Phi^{j_0-1}(y-\theta)\Phi^{m-j_0}(\theta-y)\ \phi(y-\theta)\, dy,$$

$$= \int_{-\infty}^{\infty} (v+\theta)\, C_{j_0,m}\Phi^{j_0-1}(v)\Phi^{m-j_0}(-v)\ \phi(v)dv,$$

$$= \theta \int_{-\infty}^{\infty} C_{j_0,m}\Phi^{j_0-1}(v)\Phi^{m-j_0}(-v)\ \phi(v)dv +$$

$$\int_{-\infty}^{\infty} v\, C_{j_0,m}\Phi^{j_0-1}(v)\Phi^{m-j_0}(-v)\ \phi(v)dv$$

$$= \theta + Q_{mj_0},$$

where $Q_{mj_0} = \int_{-\infty}^{\infty} v\, C_{j_0,m} \Phi^{j_0-1}(v) \Phi^{m-j_0}(-v)\ \phi(v) dv$. Hence,

$$E\left[Y_{j_0:m} - Q_{mj_0}\right] = \theta.$$

So an unbiased estimator of $\theta$ based on $Y_{j_0:m}^{(1)}, \ldots, Y_{j_0:m}^{(m)}$, $m$ independent copies of $Y_{j_0:m}$, is

$$\hat{\theta}_{b,N} = \frac{1}{m} \sum_{i=1}^{m} (Y_{j_0:m}^{(l)} - Q_{mj_0})$$

with variance given by

$$Var(\hat{\theta}_{b,N}) = \frac{1}{m}\, Var\left(Y_{j_0:m}^{(1)}\right),$$

where

$$Var\left(Y_{j_0:m}^{(1)}\right) = E\left({Y_{j_0:m}^{(1)}}^2\right) - \left(EY_{j_0:m}^{(1)}\right)^2$$

$$= \int_{-\infty}^{\infty} y^2\, C_{j_0,m} \Phi(y-\theta)^{j_0-1} \Phi(\theta - y)^{m-j_0} \phi(y-\theta) dy -$$

$$\left( \int_{-\infty}^{\infty} y\, C_{j_0,m} \Phi(y-\theta)^{j_0-1} \Phi(\theta - y)^{m-j_0} \phi(y-\theta)\, dy \right)^2,$$

$$= \int_{-\infty}^{\infty} (v+\theta)^2\, C_{j_0,m} \Phi(v)^{j_0-1} \Phi(-v)^{m-j_0} \phi(v) dv -$$

$$\left( \int_{-\infty}^{\infty} (v+\theta)\, C_{j_0,m} \Phi(v)^{j_0-1} \Phi(-v)^{m-j_0} \phi(v)\, dv \right)^2.$$

It can be noted that $Var\left(Y_{j_0:m}^{(1)}\right)$ is free of $\theta$.

Under McIntyre' s RSS scheme, an estimator of $\theta$ is given by

$$\hat{\theta}_{RSS,N} = \frac{1}{m}\sum_{j=1}^{m}(Y_{j:m} - Q_{mj})$$

with variance given by

$$Var(\hat{\theta}_{RSS,N}) = \frac{1}{m^2}\sum_{j=1}^{m}\sigma_{j:m}^2,$$

where

$$\sigma_{j:m}^2 = \int_{-\infty}^{\infty} y^2\, C_{j,m}\Phi(y-\theta)^{j-1}\Phi(\theta-y)^{m-j}\phi(y-\theta)dy -$$

$$\left(\int_{-\infty}^{\infty} y\, C_{j,m}\Phi(y-\theta)^{j-1}\Phi(\theta-y)^{m-j}\phi(y-\theta)\,dy\right)^2,$$

$$= \int_{-\infty}^{\infty}(v+\theta)^2\, C_{j,m}\Phi(v)^{j-1}\Phi(-v)^{m-j}\phi(v)dv -$$

$$\left(\int_{-\infty}^{\infty}(v+\theta)\, C_{j,m}\Phi(v)^{j-1}\Phi(-v)^{m-j}\phi(v)\,dv\right)^2.$$

Note that $\sigma_{j:m}^2$ is free of $\theta$ for all $j$. Hence the efficiency of $\hat{\theta}_{b,N}$ with respect to $\hat{\theta}_{RSS,N}$ is also free of $\theta$ and is given by

$$eff(\hat{\theta}_{RSS,N}, \hat{\theta}_{b,N}) = \frac{\sum_{j=1}^{m}\sigma_{j:m}^2}{m\,Var\left(Y_{j_0:m}^{(1)}\right)}.$$

The values of $eff(\hat{\theta}_{RSS,N}, \hat{\theta}_{b,N})$ are obtained numerically for $m$=3, …, 10 and reported in Table 3.1

Table 3.1 The efficiency of $\hat{\theta}_{b,N}$ w.r.t $\hat{\theta}_{RSS,N}$ for $N(\theta,1)$.

| $m$ | $eff(\hat{\theta}_{RSS,N},\hat{\theta}_{b,N})$ | $m$ | $eff(\hat{\theta}_{RSS,N},\hat{\theta}_{b,N})$ |
|---|---|---|---|
| 3 | 1.16463 | 7 | 1.32181 |
| 4 | 1.18208 | 8 | 1.33591 |
| 5 | 1.25853 | 9 | 1.36870 |
| 6 | 1.27494 | 10 | 1.38078 |

We note that $eff(\hat{\theta}_{RSS,N},\hat{\theta}_{b,N}) \geq 1$ for all $m$=3, …, 10 and it is increasing in $m$.

**b. Logistic(θ, 1)**

Let $(j_0, \ldots, j_0)$ be the best ranked set sampling scheme for $logistic\ (\theta,1)$ obtained in chapter two. The pdf $Y_{j_0:m}$ is

$$g_{j_0}(y;\ \theta) = \frac{C_{j_0,m}e^{-(y-\theta)(m-j_0+1)}}{(1+e^{-(y-\theta)})^{m+1}},$$

and the mean of $Y_{j_0:m}$ is

$$E(Y_{j_0:m}) = \int_{-\infty}^{\infty} y\frac{C_{j_0,m}e^{-(y-\theta)(m-j_0+1)}}{(1+e^{-(y-\theta)})^{m+1}}\,dy,$$

$$= \int_{-\infty}^{\infty} (v+\theta)\frac{C_{j_0,m}e^{-v(m-j_0+1)}}{(1+e^{-v})^{m+1}}\,dv,$$

$$= \theta\int_{-\infty}^{\infty} \frac{C_{j_0,m}e^{-v(m-j_0+1)}}{(1+e^{-v})^{m+1}}\,dv + \int_{-\infty}^{\infty} v\frac{C_{j_0,m}e^{-v(m-j_0+1)}}{(1+e^{-v})^{m+1}}\,dv,$$

$$= \theta + B_{mj_0},$$

where $B_{mj_0} = \int_{-\infty}^{\infty} v\frac{C_{j_0,m}e^{-v(m-j_0+1)}}{(1+e^{-v})^{m+1}}\,dv.$

Hence,

$$E\left(Y_{j_0:m} - B_{mj_0}\right) = \theta.$$

So an unbiased estimator of $\theta$ based on $Y_{j_0:m}^{(1)}, \ldots, Y_{j_0:m}^{(m)}$, $m$ independent copies of $Y_{j_0:m}$, is

$$\hat{\theta}_{b,L} = \frac{1}{m}\sum_{i=1}^{m}(Y_{j_0:m}^{(i)} - B_{mj_0})$$

with variance given by

$$Var(\hat{\theta}_{b,L}) = \frac{1}{m} Var\left(Y_{j_0:m}^{(1)}\right),$$

where

$$Var\left(Y_{j_0:m}^{(1)}\right) = E\left({Y_{j_0:m}^{(1)}}^2\right) - \left(EY_{j_0:m}^{(1)}\right)^2,$$

$$= \int_{-\infty}^{\infty} y^2 \frac{C_{j_0,m} e^{-(y-\theta)(m-j_0+1)}}{(1+e^{-(y-\theta)})^{m+1}} dy - \left(\int_{-\infty}^{\infty} y \frac{C_{j_0,m} e^{-(y-\theta)(m-j_0+1)}}{(1+e^{-(y-\theta)})^{m+1}} dy\right)^2,$$

$$= \int_{-\infty}^{\infty} (v+\theta)^2 \frac{C_{j_0,m} e^{-v(m-j_0+1)}}{(1+e^{-v})^{m+1}} dv - \left(\int_{-\infty}^{\infty} (v+\theta) \frac{C_{j_0,m} e^{-v(m-j_0+1)}}{(1+e^{-v})^{m+1}} dv\right)^2.$$

Under McIntyre' s RSS scheme, an estimator of $\theta$ is given by

$$\hat{\theta}_{RSS,L} = \frac{1}{m}\sum_{j=1}^{m}(Y_{j:m} - B_{mj})$$

with variance given by

$$Var(\hat{\theta}_{RSS,L}) = \frac{1}{m^2}\sum_{j=1}^{m}\sigma_{j:m}^2,$$

where

$$\sigma_{j:m}^2 = \int_{-\infty}^{\infty} y^2 \frac{C_{j,m} e^{-(y-\theta)(m-j+1)}}{(1+e^{-(y-\theta)})^{m+1}} dy - \left(\int_{-\infty}^{\infty} y \frac{C_{j,m} e^{-(y-\theta)(m-j+1)}}{(1+e^{-(y-\theta)})^{m+1}} dy\right)^2,$$

$$= \int_{-\infty}^{\infty} (v+\theta)^2 \frac{C_{j,m} e^{-v(m-j+1)}}{(1+e^{-v})^{m+1}} dv - \left(\int_{-\infty}^{\infty} (v+\theta) \frac{C_{j,m} e^{-v(m-j+1)}}{(1+e^{-v})^{m+1}} dv\right)^2.$$

The efficiency of $\hat{\theta}_{b,L}$ with respect to $\hat{\theta}_{RSS,N}$ (free of $\theta$) is

$$eff\left(\hat{\theta}_{RSS,L}, \hat{\theta}_{b,L}\right) = \frac{\sum_{j=1}^{m} \sigma_{j:m}^2}{m\, Var\left(Y_{j_0:m}^{(1)}\right)}.$$

Table 3.2 contains the numerical values of $eff\left(\hat{\theta}_{RSS,L}, \hat{\theta}_{b,L}\right)$ for $m$=3, …, 10.

Table 3.2 The efficiency of $\hat{\theta}_{b,L}$ w.r.t $\hat{\theta}_{RSS,L}$ for Logistic(θ, 1).

| $m$ | $eff\left(\hat{\theta}_{RSS,L}, \hat{\theta}_{b,L}\right)$ | $m$ | $eff\left(\hat{\theta}_{RSS,L}, \hat{\theta}_{b,L}\right)$ |
|---|---|---|---|
| 3 | 1.38137 | 7 | 1.76325 |
| 4 | 1.42027 | 8 | 1.79866 |
| 5 | 1.60667 | 9 | 1.88263 |
| 6 | 1.64601 | 10 | 1.91408 |

Similarly, $eff\left(\hat{\theta}_{RSS,L}, \hat{\theta}_{b,L}\right) \geq 1$ for all $m$=3, …, 10 and $eff\left(\hat{\theta}_{RSS,L}, \hat{\theta}_{b,L}\right)$ is increasing in $m$.

**c. LN(θ, 1)**

Let $(j_0, \dots, j_0)$ be the best ranked set sampling scheme for LN $(\theta, 1)$ obtained in chapter two.

The pdf $Y_{j_0:m}$ is

$$g_{j_0}(y;\, \theta) = \frac{C_{j_0,m}}{y\sqrt{2\pi}} \Phi^{j_0-1}(logy-\theta)\Phi^{m-j_0}(\theta-logy) e^{-\frac{1}{2}(logy-\theta)^2}.$$

and the mean of $Y_{j_0:m}$ is

$$E\left(logY_{j_0:m}\right) = \int_0^{\infty} logy \frac{C_{j_0,m}}{y\sqrt{2\pi}} \Phi^{j_0-1}(logy-\theta)\Phi^{m-j_0}(\theta-logy)e^{-\frac{1}{2}(logy-\theta)^2}\,dy,$$

$$= \int_{-\infty}^{\infty}(v+\theta)\frac{C_{j_0,m}}{\sqrt{2\pi}}\Phi^{j_0-1}(v)\Phi^{m-j_0}(-v)e^{-\frac{v^2}{2}}dv,$$

$$= \theta + \int_{-\infty}^{\infty} v\frac{C_{j_0,m}}{\sqrt{2\pi}}\Phi^{j_0-1}(v)\Phi^{m-j_0}(-v)e^{-\frac{v^2}{2}}dv,$$

$$= \theta + F_{mj_0},$$

where $F_{mj_0} = \int_{-\infty}^{\infty} v\frac{C_{j_0,m}}{\sqrt{2\pi}}\Phi^{j_0-1}(v)\Phi^{m-j_0}(-v)e^{-\frac{v^2}{2}}dv.$

Hence, $E\left(logY_{j_0:m}-F_{mj_0}\right) = \theta.$

So an unbiased estimator of $\theta$ based on $Y_{j_0:m}^{(1)},\dots,Y_{j_0:m}^{(m)}$, $m$ independent copies of $Y_{j_0:m}$, is

$$\hat{\theta}_{b,LN} = \frac{1}{m}\sum_{i=1}^{m}\left(logY_{j_0:m}^{(i)} - F_{mj_0}\right)$$

with variance given by

$$Var\left(\hat{\theta}_{b,LN}\right) = \frac{1}{m}\,Var\left(logY_{j_0:m}^{(1)}\right),$$

where

$$Var\left(logY_{j_0:m}^{(1)}\right) = E\left({logY_{j_0:m}^{(1)}}^2\right) - \left(ElogY_{j_0:m}^{(1)}\right)^2,$$

$$= \int_0^{\infty}(logy)^2\frac{C_{j_0,m}}{y\sqrt{2\pi}}\Phi^{j_0-1}(logy-\theta)\Phi^{m-j_0}(\theta-logy)e^{-\frac{1}{2}(logy-\theta)^2}\,dy -$$

$$\left(\int_0^{\infty} logy\frac{C_{j_0,m}}{y\sqrt{2\pi}}\Phi^{j_0-1}(logy-\theta)\Phi^{m-j_0}(\theta-logy)e^{-\frac{1}{2}(logy-\theta)^2}\,dy\right)^2,$$

$$= \int_{-\infty}^{\infty} (v+\theta)^2 \frac{C_{j_0,m}}{\sqrt{2\pi}} \Phi^{j_0-1}(v)\Phi^{m-j_0}(-v)e^{-\frac{v^2}{2}}\,dv -$$

$$\left( \int_{-\infty}^{\infty} (v+\theta) \frac{C_{j_0,m}}{\sqrt{2\pi}} \Phi^{j_0-1}(v)\Phi^{m-j_0}(-v)e^{-\frac{v^2}{2}}\,dv \right)^2 .$$

Under McIntyre' s RSS scheme, an estimator of $\theta$ is given by

$$\hat{\theta}_{RSS,LN} = \frac{1}{m} \sum_{j=1}^{m} logY_{j:m}$$

with variance given by

$$Var\left(\hat{\theta}_{RSS,LN}\right) = \frac{1}{m^2} \sum_{j=1}^{m} \sigma_{j:m}^2,$$

where

$$\sigma_{j:m}^2 = \int_{0}^{\infty} (logy)^2 \frac{C_{j,m}}{y\sqrt{2\pi}} \Phi^{j-1}(logy-\theta)\Phi^{m-j}(\theta-logy)e^{-\frac{1}{2}(logy-\theta)^2}\,dy -$$

$$\left( \int_{0}^{\infty} logy \frac{C_{j,m}}{y\sqrt{2\pi}} \Phi^{j-1}(logy-\theta)\Phi^{m-j}(\theta-logy)e^{-\frac{1}{2}(logy-\theta)^2}\,dy \right)^2 ,$$

$$= \int_{-\infty}^{\infty} (v+\theta)^2 \frac{C_{j,m}}{\sqrt{2\pi}} \Phi^{j-1}(v)\Phi^{m-j}(-v)e^{-\frac{v^2}{2}}\,dv -$$

$$\left( \int_{-\infty}^{\infty} (v+\theta) \frac{C_{j,m}}{\sqrt{2\pi}} \Phi^{j-1}(v)\Phi^{m-j}(-v)e^{-\frac{v^2}{2}}\,dv \right)^2 .$$

The efficiency of $\hat{\theta}_{b,LN}$ with respect to $\hat{\theta}_{RSS,LN}$

$$eff\left(\hat{\theta}_{RSS,LN}, \hat{\theta}_{b,LN}\right) = \frac{\sum_{j=1}^{m} \sigma_{j:m}^2}{m\, Var\left(log Y_{j_0:m}^{(1)}\right)}.$$

We note that $eff\left(\hat{\theta}_{RSS,LN}, \hat{\theta}_{b,LN}\right) = eff\left(\hat{\theta}_{RSS,N}, \hat{\theta}_{b,N}\right)$, which is expected since $\hat{\theta}_{b,LN}$ is equivalent to $\hat{\theta}_{b,N}$.

**d. Cauchy $(\theta, 1)$**

In chapter two, we showed that the best sampling schemes, when estimating the mode of the Cauchy distribution , depends on a middle order statistic. Since Cauchy$(\theta, 1)$ has no mean, we may compare the estimators, obtained via the best sampling schemes, to their counterparts through the MLE method. So we leave this to section 3.4.

## 3.2 Estimators for scale distribution

**a. Exp$(\theta)$**

Let $(j_0, \dots, j_0)$ be the best ranked set sampling scheme for exp($\theta$ ) obtained in chapter two.

The pdf of $Y_{j_0:m}$ is

$$g_{j_0}(y; \theta) = C_{j_0,m}\left(1 - e^{-\frac{y}{\theta}}\right)^{j_0-1} e^{-(m-j_0+1)\frac{y}{\theta}} \frac{1}{\theta}; \qquad y > 0,$$

where $C_{j_0,m} = \frac{m!}{(j_0-1)!(m-j_0)!}$. The mean of $Y_{j_0:m}$ is

$$E\left(Y_{j_0:m}\right) = C_{j_0,m} \int_0^\infty \frac{y}{\theta}\left(1 - e^{-\frac{y}{\theta}}\right)^{j_0-1} e^{-(m-j_0+1)\frac{y}{\theta}} dy,$$

$$= \theta\, A_{mj_0},$$

where

$$A_{mj_0} = C_{j_0,m} \int_0^\infty v\,(1 - e^{-v})^{j_0-1} e^{-(m-j_0+1)v}\, dv.$$

An unbiased estimator of $\theta$ based on $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \ldots, Y_{j_0:m}^{(m)}$, $m$ independent copies of $Y_{j_0:m}$, is

$$\hat{\theta}_{b,e} = \frac{1}{mA_{mj_0}} \sum_{i=1}^{m} Y_{j_0:m}^{(i)}.$$

The variance of $\hat{\theta}_{b,e}$ is

$$Var(\hat{\theta}_{b,e}) = \frac{1}{mA_{mj_0}^2} Var\left(Y_{j_0:m}^{(1)}\right),$$

where

$$Var\left(Y_{j_0:m}^{(1)}\right) = E\left({Y_{j_0:m}^{(1)}}^2\right) - \left(EY_{j_0:m}^{(1)}\right)^2,$$

$$= C_{j_0,m} \int_0^\infty \theta^2 v^2 (1 - e^{-v})^{j_0-1} e^{-(m-j_0+1)v}\, dv -$$

$$\left(C_{j_0,m} \int_0^\infty \theta v\,(1 - e^{-v})^{j_0-1} e^{-(m-j_0+1)v}\, dv\right)^2.$$

Under the McIntyre's RSS, the estimator

$$\hat{\theta}_{RSS,e} = \frac{1}{m} \sum_{j=1}^{m} \frac{Y_{j:m}}{A_{mj}}$$

is unbiased for $\theta$ and has the variance

$$Var(\hat{\theta}_{RSS,e}) = \frac{1}{m^2} \sum_{j=1}^{m} \frac{\sigma_{j:m}^2}{A_{mj}^2},$$

where

$$\sigma_{j:m}^2 = \int_0^\infty y^2\, C_{j,m}\left(1-e^{-\frac{y}{\theta}}\right)^{j-1} e^{-(m-j+1)\frac{y}{\theta}}\,\frac{1}{\theta}\,dy - \left(\int_0^\infty y\, C_{j,m}\left(1-e^{-\frac{y}{\theta}}\right)^{j-1} e^{-(m-j+1)\frac{y}{\theta}}\,\frac{1}{\theta}\,dy\right)^2,$$

$$= C_{j,m}\int_0^\infty \theta^2 v^2\,(1-e^{-v})^{j-1} e^{-(m-j+1)v}\,dv - \left(C_{j,m}\int_0^\infty \theta v\,(1-e^{-v})^{j-1} e^{-(m-j+1)v}\,dv\right)^2.$$

The efficiency of $\hat{\theta}_{b,e}$ with respect to $\hat{\theta}_{RSS,e}$

$$eff\left(\hat{\theta}_{RSS,e},\hat{\theta}_{b,e}\right) = \frac{A_{mj_0}^2\sum_{j=1}^{m}\left(\sigma_{j:m}^2/A_{mj}^2\right)}{m\,Var\left(Y_{j_0:m}^{(1)}\right)}.$$

Table 3.3 The efficiency of $\hat{\theta}_{b,e}$ w.r.t $\hat{\theta}_{RSS,e}$ for Exp$(\theta)$.

| $m$ | $eff\left(\hat{\theta}_{RSS,e},\hat{\theta}_{b,e}\right)$ | $m$ | $eff\left(\hat{\theta}_{RSS,e},\hat{\theta}_{b,e}\right)$ |
|---|---|---|---|
| 3 | 1.58449 | 7 | 1.9473 |
| 4 | 1.67619 | 8 | 2.01647 |
| 5 | 1.72161 | 9 | 2.06672 |
| 6 | 1.8515 | 10 | 2.10317 |

We note from the above table that the $eff\left(\hat{\theta}_{RSS,e},\hat{\theta}_{b,e}\right) \geq 1$ for all $m$=3, …, 10 and $eff\left(\hat{\theta}_{RSS,e},\hat{\theta}_{b,e}\right)$ is increasing in $m$.

**b. $N(0,\sigma^2)$**

Let $(j_0, \dots, j_0)$ be the best ranked set sampling scheme for $N(0,\sigma^2)$ obtained in chapter two.

The pdf $Y_{j_0:m}$ is

$$g_{j_0}(y;\,\sigma) = \frac{1}{\sigma} C_{j_0,m}\,\Phi^{j_0-1}\left(\frac{y}{\sigma}\right)\;\Phi^{m-j_0}\left(-\frac{y}{\sigma}\right)\phi\left(\frac{y}{\sigma}\right),$$

and the mean of $Y_{j_0:m}$ is

$$E\left(Y_{j_0:m}\right) = \int_{-\infty}^{\infty} \frac{y}{\sigma} C_{j_0,m}\, \Phi^{j_0-1}\left(\frac{y}{\sigma}\right)\ \Phi^{m-j_0}\left(-\frac{y}{\sigma}\right) \phi\left(\frac{y}{\sigma}\right) dy,$$

$$= \int_{-\infty}^{\infty} v\sigma\, C_{j_0,m} \Phi^{j_0-1}(v) \Phi^{m-j_0}(-v)\ \ \phi(v) dv,$$

$$= \sigma \int_{-\infty}^{\infty} v C_{j_0,m} \Phi^{j_0-1}(v) \Phi^{m-j_0}(-v)\ \ \phi(v) dv,$$

$$= \sigma U_{mj_0},$$

where $U_{mj_0} = \int_{-\infty}^{\infty} v\, C_{j_0,m} \Phi^{j_0-1}(v) \Phi^{m-j_0}(-v)\ \ \phi(v) dv$. Hence,

$$E\left(\frac{Y_{j_0:m}}{U_{mj_0}}\right) = \sigma.$$

So an unbiased estimator of $\sigma$ based on $Y_{j_0:m}^{(1)}, \dots, Y_{j_0:m}^{(m)}$, $m$ independent copies of $Y_{j_0:m}$, is

$$\hat{\sigma}_{b,N} = \frac{1}{m} \sum_{i=1}^{m} \left(\frac{Y_{j_0:m}^{(i)}}{U_{mj_0}}\right)$$

with variance given by

$$Var\left(\hat{\sigma}_{b,N}\right) = \frac{1}{mU_{mj_0}^2}\ Var\left(Y_{j_0:m}^{(1)}\right),$$

where

$$Var\left(Y_{j_0:m}^{(1)}\right) = E\left({Y_{j_0:m}^{(1)}}^2\right) - \left(EY_{j_0:m}^{(1)}\right)^2,$$

$$= \int_{-\infty}^{\infty} y^2 \frac{1}{\sigma} C_{j_0,m}\, \Phi^{j_0-1}\left(\frac{y}{\sigma}\right)\ \Phi^{m-j_0}\left(-\frac{y}{\sigma}\right) \phi\left(\frac{y}{\sigma}\right) dy -$$

$$\left(\int_{-\infty}^{\infty} \frac{y}{\sigma} C_{j_0,m}\, \Phi^{j_0-1}\left(\frac{y}{\sigma}\right)\ \Phi^{m-j_0}\left(-\frac{y}{\sigma}\right) \phi\left(\frac{y}{\sigma}\right) dy\right)^2,$$

$$= \int_{-\infty}^{\infty} \sigma^2 v^2 C_{j_0,m} \Phi^{j_0-1}(v) \ \Phi^{m-j_0}(-v)\phi(v)dv -$$

$$\left(\int_{-\infty}^{\infty} v\sigma C_{j_0,m} \Phi^{j_0-1}(v) \ \Phi^{m-j_0}(-v)\phi(v)dv\right)^2.$$

Under McIntyre' s RSS scheme, an estimator of $\sigma$ is given by

$$\hat{\sigma}_{RSS,N} = \frac{1}{m}\sum_{j=1}^{m} \frac{Y_{j:m}}{U_{mj}}$$

with variance given by

$$Var(\hat{\sigma}_{RSS,N}) = \frac{1}{m^2}\sum_{j=1}^{m} \frac{\sigma_{j:m}^2}{U_{mj}^2},$$

where

$$\sigma_{j:m}^2 = \int_{-\infty}^{\infty} y^2 \frac{1}{\sigma} C_{j,m} \Phi^{j-1}\left(\frac{y}{\sigma}\right) \ \Phi^{m-j}\left(-\frac{y}{\sigma}\right) \phi\left(\frac{y}{\sigma}\right) dy -$$

$$\left(\int_{-\infty}^{\infty} y \frac{1}{\sigma} C_{j,m} \Phi^{j-1}\left(\frac{y}{\sigma}\right) \ \Phi^{m-j}\left(-\frac{y}{\sigma}\right) \phi\left(\frac{y}{\sigma}\right) dy -\right)^2,$$

$$= \int_{-\infty}^{\infty} \sigma^2 v^2 C_{j,m} \Phi^{j-1}(v) \ \Phi^{m-j}(-v)\phi(v)dv -$$

$$\left(\int_{-\infty}^{\infty} v\sigma C_{j,m} \Phi^{j-1}(v) \ \Phi^{m-j}(-v)\phi(v)dv\right)^2.$$

The efficiency of $\hat{\sigma}_{b,N}$ with respect to $\hat{\sigma}_{RSS,N}$

$$eff\left(\hat{\sigma}_{RSS,N}, \hat{\sigma}_{b,N}\right) = \frac{U_{mj_0}^2 \sum_{j=1}^{m}\left(\sigma_{j:m}^2 / U_{mj}^2\right)}{m\, Var\left(Y_{j_0:m}^{(1)}\right)}.$$

Table 3.4 The efficiency of $\hat{\sigma}_{b,N}$ w.r.t $\hat{\sigma}_{RSS,N}$ for $N(0, \sigma^2)$ .

| $m$ | $eff\left(\hat{\sigma}_{RSS,N}, \hat{\sigma}_{b,N}\right)$ | $m$ | $eff\left(\hat{\sigma}_{RSS,N}, \hat{\sigma}_{b,N}\right)$ |
|---|---|---|---|
| 3 | 1 | 7 | 3.77595 |
| 4 | 4.90272 | 8 | 12.8528 |
| 5 | 2.42098 | 9 | 5.02102 |
| 6 | 9.00769 | 10 | 16.397 |

We note from the above table that the $eff\left(\hat{\sigma}_{RSS,N}, \hat{\sigma}_{b,N}\right) \geq 1$ for all $m$=3, …, 10 .

**c. Logistic (0, $\sigma$)**

Let $(j_0, \ldots, j_0)$ be the best ranked set sampling scheme for Logistic$(0, \sigma\ )$ obtained in chapter two. The pdf $Y_{j_0:m}$ is

$$g_{j_0}(y;\ \sigma) = \frac{C_{j_0,m} e^{-\frac{y(m-j_0+1)}{\sigma}}}{\sigma\left(1 + e^{-\left(\frac{y}{\sigma}\right)}\right)^{m+1}},$$

and the mean of $Y_{j_0:m}$ is

$$E\left(Y_{j_0:m}\right) = \int_{-\infty}^{\infty} y \frac{C_{j_0,m} e^{-\frac{y(m-j_0+1)}{\sigma}}}{\sigma\left(1 + e^{-\left(\frac{y}{\sigma}\right)}\right)^{m+1}}\, dy,$$

$$= \int_{-\infty}^{\infty} v\sigma \frac{C_{j_0,m} e^{-v(m-j_0+1)}}{(1 + e^{-v})^{m+1}}\, dv,$$

$$= \sigma \int_{-\infty}^{\infty} v \frac{C_{j_0,m} e^{-v(m-j_0+1)}}{(1 + e^{-v})^{m+1}}\, dv,$$

$$= \sigma L_{mj_0},$$

where $L_{mj_0} = \int_{-\infty}^{\infty} v \frac{C_{j_0,m} e^{-v(m-j_0+1)}}{(1+e^{-v})^{m+1}}\, dv$ . Hence,

$$E\left(\frac{Y_{j_0:m}}{L_{mj_0}}\right) = \sigma.$$

So an unbiased estimator of $\sigma$ based on $Y_{j_0:m}^{(1)}, \ldots, Y_{j_0:m}^{(m)}$, $m$ independent copies of $Y_{j_0:m}$, is

$$\hat{\sigma}_{b,L} = \frac{1}{m}\sum_{i=1}^{m}\left(\frac{Y_{j_0:m}^{(i)}}{L_{mj_0}}\right)$$

with variance given by

$$Var(\hat{\sigma}_{b,L}) = \frac{1}{mL_{mj_0}^2} Var\left(Y_{j_0:m}^{(1)}\right),$$

where

$$Var\left(Y_{j_0:m}^{(1)}\right) = E\left({Y_{j_0:m}^{(1)}}^2\right) - \left(EY_{j_0:m}^{(1)}\right)^2,$$

$$= \int_{-\infty}^{\infty} y^2 \frac{C_{j_0,m} e^{-\frac{y(m-j_0+1)}{\sigma}}}{\sigma\left(1+e^{-\left(\frac{y}{\sigma}\right)}\right)^{m+1}} dy - \left(\int_{-\infty}^{\infty} y \frac{C_{j_0,m} e^{-\frac{y(m-j_0+1)}{\sigma}}}{\sigma\left(1+e^{-\left(\frac{y}{\sigma}\right)}\right)^{m+1}} dy\right)^2,$$

$$= \int_{-\infty}^{\infty} \sigma^2 v^2 \frac{C_{j_0,m} e^{-v(m-j_0+1)}}{(1+e^{-v})^{m+1}} dv - \left(\int_{-\infty}^{\infty} v\sigma \frac{C_{j_0,m} e^{-v(m-j_0+1)}}{(1+e^{-v})^{m+1}} dv\right)^2.$$

Under McIntyre' s RSS scheme, an estimator of $\sigma$ is given by

$$\hat{\sigma}_{RSS,L} = \frac{1}{m}\sum_{j=1}^{m}\frac{Y_{j:m}}{L_{mj}}$$

with variance given by

$$Var(\hat{\sigma}_{RSS,L}) = \frac{1}{m^2}\sum_{j=1}^{m}\frac{\sigma_{j:m}^2}{L_{mj}^2},$$

where

$$\sigma_{j:m}^2 = \int_{-\infty}^{\infty} y^2 \frac{C_{j_0,m} e^{-\frac{y(m-j_0+1)}{\sigma}}}{\sigma\left(1+e^{-\left(\frac{y}{\sigma}\right)}\right)^{m+1}} dy - \left(\int_{-\infty}^{\infty} y \frac{C_{j_0,m} e^{-\frac{y(m-j_0+1)}{\sigma}}}{\sigma\left(1+e^{-\left(\frac{y}{\sigma}\right)}\right)^{m+1}} dy\right)^2,$$

$$= \int_{-\infty}^{\infty} \sigma^2 v^2 \frac{C_{j,m} e^{-v(m-j+1)}}{(1+e^{-v})^{m+1}} dv - \left(\int_{-\infty}^{\infty} v\sigma \frac{C_{j,m} e^{-v(m-j+1)}}{(1+e^{-v})^{m+1}} dv\right)^2.$$

The efficiency of $\hat{\sigma}_{b,N}$ with respect to $\hat{\sigma}_{RSS,N}$

$$eff\left(\hat{\sigma}_{RSS,L}, \hat{\sigma}_{b,L}\right) = \frac{L_{mj_0}^2 \sum_{j=1}^{m}\left(\sigma_{j:m}^2 / L_{mj}^2\right)}{m\,Var\left(Y_{j_0:m}^{(1)}\right)}.$$

Table 3.5 The efficiency of $\hat{\sigma}_{b,L}$ w.r.t $\hat{\sigma}_{RSS,L}$ for Logistic $(0, \sigma)$ .

| $m$ | $eff\left(\hat{\sigma}_{RSS,L}, \hat{\sigma}_{b,L}\right)$ | $m$ | $eff\left(\hat{\sigma}_{RSS,L}, \hat{\sigma}_{b,L}\right)$ |
|---|---|---|---|
| 3 | 1 | 7 | 2.90628 |
| 4 | 4.12421 | 8 | 9.13655 |
| 5 | 2.05518 | 9 | 3.61491 |
| 6 | 6.84888 | 10 | 11.099 |

We note from the above table that the $eff\left(\hat{\sigma}_{RSS,L}, \hat{\sigma}_{b,L}\right) \geq 1$ for all $m$=3, …, 10 .

**d. LN $(0, \sigma^2)$**

Let $(j_0, \ldots, j_0)$ be the best ranked set sampling scheme for Log-Normal $(0, \sigma^2)$ obtained in chapter two. The pdf $Y_{j_0:m}$ is

$$g_{j_0}(y; \sigma) = \frac{C_{j_0,m}}{y\sigma\sqrt{2\pi}} \Phi^{j_0-1}\left(\frac{logy}{\sigma}\right) \Phi^{m-j_0}\left(\frac{-logy}{\sigma}\right) e^{-\frac{1}{2\sigma^2}(logy)^2}$$

and the mean of $Y_{j_0:m}$ is

$$E\left(logY_{j_0:m}\right) = \int_{0}^{\infty} logy \frac{C_{j_0,m}}{y\sigma\sqrt{2\pi}} \Phi^{j_0-1}\left(\frac{logy}{\sigma}\right) \Phi^{m-j_0}\left(\frac{-logy}{\sigma}\right) e^{-\frac{1}{2\sigma^2}(logy)^2} dy,$$

$$= \int_{-\infty}^{\infty} v\sigma \frac{C_{j_0,m}}{\sqrt{2\pi}} \Phi^{j_0-1}(v)\Phi^{m-j_0}(-v)e^{-\frac{v^2}{2}}dv,$$

$$= \sigma \int_{-\infty}^{\infty} v \frac{C_{j_0,m}}{\sqrt{2\pi}} \Phi^{j_0-1}(v)\Phi^{m-j_0}(-v)e^{-\frac{v^2}{2}}dv,$$

$$= \sigma E_{mj_0},$$

where

$$E_{mj_0} = \int_{-\infty}^{\infty} v \frac{C_{j_0,m}}{\sqrt{2\pi}} \Phi^{j_0-1}(v)\Phi^{m-j_0}(-v)e^{-\frac{v^2}{2}}dv.$$

An unbiased estimator of $\sigma$ based on $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \ldots, Y_{j_0:m}^{(m)}$, $m$ independent copies of $Y_{j_0:m}$, is

$$\hat{\sigma}_{b,LN} = \frac{1}{mE_{mj_0}} \sum_{j=1}^{m} logY_{j_0:m}^{(j)}.$$

The variance of $\hat{\sigma}_{b,LN}$ is

$$Var(\hat{\sigma}_{b,LN}) = \frac{1}{mE_{mj_0}^2} Var\left(logY_{j_0:m}^{(1)}\right),$$

where

$$Var\left(logY_{j_0:m}^{(1)}\right) = E\left(logY_{j_0:m}^{(1)\,2}\right) - \left(E\left(logY_{j_0:m}^{(1)}\right)\right)^2,$$

$$= \int_{-\infty}^{\infty} v^2 \frac{C_{j_0,m}}{\sqrt{2\pi}} \Phi^{j_0-1}(v)\Phi^{m-j_0}(-v)e^{-\frac{v^2}{2}}dv - \left(\int_{-\infty}^{\infty} v \frac{C_{j_0,m}}{\sqrt{2\pi}} \Phi^{j_0-1}(v)\Phi^{m-j_0}(-v)e^{-\frac{v^2}{2}}dv\right)^2.$$

Under the McIntyre's RSS, the estimator

$$\hat{\sigma}_{RSS,LN} = \frac{1}{m}\sum_{j=1}^{m}\frac{logY_{j:m}}{E_{mj}}$$

is unbiased for $\sigma$ and has the variance

$$Var(\hat{\sigma}_{RSS,LN}) = \frac{1}{m^2}\sum_{i=1}^{m}\frac{\sigma_{j:m}^2}{E_{mj}^2}$$

where

$$\sigma_{j:m}^2 = \int_{-\infty}^{\infty} v^2 \frac{C_{j,m}}{\sqrt{2\pi}}\Phi^{j-1}(v)\Phi^{m-j}(-v)e^{-\frac{v^2}{2}}dv - \left(\int_{-\infty}^{\infty} v\frac{C_{j,m}}{\sqrt{2\pi}}\Phi^{j-1}(v)\Phi^{m-j}(-v)e^{-\frac{v^2}{2}}dv\right)^2.$$

The efficiency of $\hat{\sigma}_{b,LN}$ with respect to $\hat{\sigma}_{RSS,LN}$

$$eff(\hat{\sigma}_{RSS,LN},\hat{\sigma}_{b,LN}) = \frac{E_{mj_0}^2\sum_{j=1}^{m}(\sigma_{j:m}^2/E_{mj}^2)}{m\,Var\left(logY_{j_0:m}^{(1)}\right)}.$$

It is straight forward to show that $eff(\hat{\sigma}_{RSS,LN},\hat{\sigma}_{b,LN}) = eff(\hat{\sigma}_{RSS,N},\hat{\sigma}_{b,N})$ for all $m$. Hence we don't present the values of $eff(\hat{\sigma}_{RSS,LN},\hat{\sigma}_{b,LN})$ in this case.

### 3.3 Estimators for location- scale distributions with known coefficient of variation

**a. $N(\theta,\theta^2)$**

Let $(j_0,\dots,j_0)$ be the best ranked set sampling scheme for $N(\theta,\theta^2)$ obtained in chapter two. The pdf $Y_{j_0:m}$ is

$$g_{j_0}(y;\sigma) = C_{j_0,m}\Phi^{j_0-1}\left(\frac{y-\theta}{\theta}\right)\Phi^{m-j_0}\left(\frac{\theta-y}{\theta}\right)\phi\left(\frac{y-\theta}{\theta}\right)\frac{1}{\theta},$$

and the mean of $Y_{j_0:m}$ is

$$E\left(Y_{j_0:m}\right) = \int_{-\infty}^{\infty} y\, C_{j_0,m} \Phi^{j_0-1}\left(\frac{y-\theta}{\theta}\right) \Phi^{m-j_0}\left(\frac{\theta-y}{\theta}\right) \phi\left(\frac{y-\theta}{\theta}\right) \frac{1}{\theta}\, dy,$$

$$= \int_{-\infty}^{\infty} \theta(v+1) C_{j_0,m} \Phi^{j_0-1}(v) \Phi^{m-j_0}(-v) \phi(v)\, dv,$$

$$= \theta \int_{-\infty}^{\infty} (v+1) C_{j_0,m} \Phi^{j_0-1}(v) \Phi^{m-j_0}(-v) \phi(v)\, dv,$$

$$= \theta H_{mj_0},$$

where $H_{mj_0} = \int_{-\infty}^{\infty} (v+1) C_{j_0,m} \Phi^{j_0-1}(v) \Phi^{m-j_0}(-v) \phi(v)\, dv$. Hence, $E\left(\frac{Y_{j_0:m}}{H_{mj_0}}\right) = \theta$.

So an unbiased estimator of $\theta$ based on $Y_{j_0:m}^{(1)}, \dots, Y_{j_0:m}^{(m)}$, $m$ independent copies of $Y_{j_0:m}$, is

$$\hat{\theta}_{b,N} = \frac{1}{m} \sum_{i=1}^{m} \left(\frac{Y_{j_0:m}^{(i)}}{H_{mj_0}}\right)$$

with variance given by

$$Var\left(\hat{\theta}_{b,N}\right) = \frac{1}{m H_{mj_0}^2} Var\left(Y_{j_0:m}^{(1)}\right),$$

where

$$Var\left(Y_{j_0:m}^{(1)}\right) = E\left({Y_{j_0:m}^{(1)}}^2\right) - \left(E Y_{j_0:m}^{(1)}\right)^2,$$

$$= \int_{-\infty}^{\infty} y^2\, C_{j_0,m} \Phi^{j_0-1}\left(\frac{y-\theta}{\theta}\right) \Phi^{m-j_0}\left(\frac{\theta-y}{\theta}\right) \phi\left(\frac{y-\theta}{\theta}\right) \frac{1}{\theta} dy -$$

$$\left(\int_{-\infty}^{\infty} y\, C_{j_0,m} \Phi^{j_0-1}\left(\frac{y-\theta}{\theta}\right) \Phi^{m-j_0}\left(\frac{\theta-y}{\theta}\right) \phi\left(\frac{y-\theta}{\theta}\right) \frac{1}{\theta}\, dy\right)^2,$$

$$= \int_{-\infty}^{\infty} \theta^2(v+1)^2 C_{j_0,m}\Phi^{j_0-1}(v)\Phi^{m-j_0}(-v)\phi(v)\,dv -$$

$$\left(\int_{-\infty}^{\infty} \theta(v+1)\, C_{j_0,m}\Phi^{j_0-1}(v)\Phi^{m-j_0}(-v)\phi(v)\,dv\right)^2.$$

Under McIntyre' s RSS scheme, an estimator of $\theta$ is given by

$$\hat{\theta}_{RSS,N} = \frac{1}{m}\sum_{j=1}^{m}\frac{Y_{j:m}}{H_{mj}}$$

with variance given by

$$Var(\hat{\theta}_{RSS,N}) = \frac{1}{m^2}\sum_{j=1}^{m}\frac{\sigma_{j:m}^2}{H_{mj}^2},$$

where

$$\sigma_{j:m}^2 = \int_{-\infty}^{\infty} y^2\, C_{j,m}\Phi^{j-1}\left(\frac{y-\theta}{\theta}\right)\Phi^{m-j}\left(\frac{\theta-y}{\theta}\right)\phi\left(\frac{y-\theta}{\theta}\right)\frac{1}{\theta}\,dy -$$

$$\left(\int_{-\infty}^{\infty} y\, C_{j,m}\Phi^{j-1}\left(\frac{y-\theta}{\theta}\right)\Phi^{m-j}\left(\frac{\theta-y}{\theta}\right)\phi\left(\frac{y-\theta}{\theta}\right)\frac{1}{\theta}\,dy\right)^2,$$

$$= \int_{-\infty}^{\infty} \theta^2(v+1)^2 C_{j,m}\Phi^{j-1}(v)\Phi^{m-j}(-v)\phi(v)\,dv -$$

$$\left(\int_{-\infty}^{\infty} \theta(v+1)\, C_{j,m}\Phi^{j-1}(v)\Phi^{m-j}(-v)\phi(v)\,dv\right)^2.$$

The efficiency of $\hat{\theta}_{b,N}$ with respect to $\hat{\theta}_{RSS,N}$

$$eff(\hat{\theta}_{RSS,N}, \hat{\theta}_{b,N}) = \frac{H^2_{mj_0} \sum_{j=1}^{m} (\sigma^2_{j:m}/H^2_{mj})}{m\,Var\left(Y^{(1)}_{j_0:m}\right)}.$$

Table 3.6 The efficiency of $\hat{\theta}_{b,N}$ w.r.t $\hat{\theta}_{RSS,N}$ for $N(\theta, \theta^2)$.

| $m$ | $eff(\hat{\theta}_{RSS,N}, \hat{\theta}_{b,N})$ | $m$ | $eff(\hat{\theta}_{RSS,N}, \hat{\theta}_{b,N})$ |
|---|---|---|---|
| 3 | 3.18374 | 7 | 3.92696 |
| 4 | 3.56868 | 8 | 3.93897 |
| 5 | 3.77368 | 9 | 3.92863 |
| 6 | 3.87940 | 10 | 3.89431 |

We note from the above table that the $eff(\hat{\theta}_{RSS,N}, \hat{\theta}_{b,N}) \geq 1$ for all $m$=3, ..., 10.

**b. Logistic distribution with known c.v.**

Let $(j_0, \dots, j_0)$ be the best ranked set sampling scheme for Logistic$(\theta, \sigma)$ obtained in chapter two. The pdf $Y_{j_0:m}$ is

$$g_{j_0}(y;\, \theta) = \frac{\pi C_{j_0,m} e^{\frac{\pi(y-\theta)(m-j_0+1)}{c\sqrt{3}\theta}}}{c\sqrt{3}\theta\left(1 + e^{-\frac{(y-\theta)\pi}{\theta c\sqrt{3}}}\right)^{m+1}}$$

and the mean of $Y_{j_0:m}$ is

$$E\left(Y_{j_0:m}\right) = \int_{-\infty}^{\infty} y \frac{\pi C_{j_0,m} e^{-\frac{\pi(y-\theta)(m-j_0+1)}{c\sqrt{3}\theta}}}{c\sqrt{3}\theta\left(1 + e^{-\frac{(y-\theta)\pi}{\theta c\sqrt{3}}}\right)^{m+1}}\, dy,$$

$$= \int_{-\infty}^{\infty} \theta(v+1) \frac{\pi C_{j_0,m} e^{-\frac{\pi v(m-j_0+1)}{c\sqrt{3}}}}{c\sqrt{3}\left(1 + e^{-\frac{v\pi}{c\sqrt{3}}}\right)^{m+1}}\, dv,$$

$$= \theta \int_{-\infty}^{\infty} (v+1) \frac{\pi C_{j_0,m} e^{-\frac{\pi v(m-j_0+1)}{c\sqrt{3}}}}{c\sqrt{3}\left(1+e^{-\frac{v\pi}{c\sqrt{3}}}\right)^{m+1}} dv,$$

$$= \theta D_{mj_0},$$

where $D_{mj_0} = \int_{-\infty}^{\infty} (v+1) \frac{\pi C_{j_0,m} e^{-\frac{\pi v(m-j_0+1)}{c\sqrt{3}}}}{c\sqrt{3}\left(1+e^{-\frac{v\pi}{c\sqrt{3}}}\right)^{m+1}} dv$. Hence,

$$E\left(\frac{Y_{j_0:m}}{D_{mj_0}}\right) = \theta.$$

So an unbiased estimator of $\theta$ based on $Y_{j_0:m}^{(1)}, \dots, Y_{j_0:m}^{(m)}$, $m$ independent copies of $Y_{j_0:m}$, is

$$\hat{\theta}_{b,L} = \frac{1}{m}\sum_{i=1}^{m}\left(\frac{Y_{j_0:m}^{(i)}}{D_{mj_0}}\right)$$

with variance given by

$$Var(\hat{\theta}_{b,L}) = \frac{1}{mD_{mj_0}^2} Var\left(Y_{j_0:m}^{(1)}\right),$$

where

$$Var\left(Y_{j_0:m}^{(1)}\right) = E\left({Y_{j_0:m}^{(1)}}^2\right) - \left(EY_{j_0:m}^{(1)}\right)^2,$$

$$= \int_{-\infty}^{\infty} y^2 \frac{\pi C_{j_0,m} e^{-\frac{\pi(y-\theta)(m-j_0+1)}{c\sqrt{3}\theta}}}{c\sqrt{3}\theta\left(1+e^{-\frac{(y-\theta)\pi}{\theta c\sqrt{3}}}\right)^{m+1}} dy - \left(\int_{-\infty}^{\infty} y \frac{\pi C_{j_0,m} e^{-\frac{\pi(y-\theta)(m-j_0+1)}{c\sqrt{3}\theta}}}{c\sqrt{3}\theta\left(1+e^{-\frac{(y-\theta)\pi}{\theta c\sqrt{3}}}\right)^{m+1}} dy\right)^2,$$

$$= \int_{-\infty}^{\infty} \theta^2(v+1)^2 \frac{\pi C_{j_0,m} e^{-\frac{\pi v(m-j_0+1)}{c\sqrt{3}}}}{c\sqrt{3}\left(1+e^{-\frac{v\pi}{c\sqrt{3}}}\right)^{m+1}} dv - \left( \int_{-\infty}^{\infty} \theta(v+1) \frac{\pi C_{j_0,m} e^{-\frac{\pi v(m-j_0+1)}{c\sqrt{3}}}}{c\sqrt{3}\left(1+e^{-\frac{v\pi}{c\sqrt{3}}}\right)^{m+1}} dv \right)^2.$$

Under McIntyre' s RSS scheme, an estimator of $\theta$ is given by

$$\hat{\theta}_{RSS,L} = \frac{1}{m}\sum_{j=1}^{m} \frac{Y_{j:m}}{D_{mj}}$$

with variance given by

$$Var(\hat{\theta}_{RSS,L}) = \frac{1}{m^2}\sum_{j=1}^{m} \frac{\sigma_{j:m}^2}{D_{mj}^2},$$

where

$$\sigma_{j:m}^2 = \int_{-\infty}^{\infty} y^2 \frac{\pi C_{j,m} e^{-\frac{\pi(y-\theta)(m-j+1)}{c\sqrt{3}\theta}}}{c\sqrt{3}\theta\left(1+e^{-\frac{(y-\theta)\pi}{\theta c\sqrt{3}}}\right)^{m+1}} dy - \left( \int_{-\infty}^{\infty} y \frac{\pi C_{j,m} e^{-\frac{\pi(y-\theta)(m-j+1)}{c\sqrt{3}\theta}}}{c\sqrt{3}\theta\left(1+e^{-\frac{(y-\theta)\pi}{\theta c\sqrt{3}}}\right)^{m+1}} dy \right)^2,$$

$$= \int_{-\infty}^{\infty} \theta^2(v+1)^2 \frac{\pi C_{j,m} e^{-\frac{\pi v(m-j+1)}{c\sqrt{3}}}}{c\sqrt{3}\left(1+e^{-\frac{v\pi}{c\sqrt{3}}}\right)^{m+1}} dv - \left( \int_{-\infty}^{\infty} \theta(v+1) \frac{\pi C_{j,m} e^{-\frac{\pi v(m-j+1)}{c\sqrt{3}}}}{c\sqrt{3}\left(1+e^{-\frac{v\pi}{c\sqrt{3}}}\right)^{m+1}} dv \right)^2.$$

The efficiency of $\hat{\theta}_{b,L}$ with respect to $\hat{\theta}_{RSS,L}$ is

$$eff(\hat{\theta}_{RSS,L}, \hat{\theta}_{b,L}) = \frac{D_{mj_0}^2 \sum_{j=1}^{m}\left(\sigma_{j:m}^2 / D_{mj}^2\right)}{m\, Var\left(Y_{j_0:m}^{(1)}\right)}.$$

Table 3.7 The efficiency of $\hat{\theta}_{b,L}$ w.r.t $\hat{\theta}_{RSS,L}$ for Logistic with known c.v .

| $m$ | $eff(\hat{\theta}_{RSS,L},\hat{\theta}_{b,L})$ | $m$ | $eff(\hat{\theta}_{RSS,L},\hat{\theta}_{b,L})$ |
|---|---|---|---|
| 3 | 2.92882 | 7 | 3.55381 |
| 4 | 3.11152 | 8 | 3.70837 |
| 5 | 3.15634 | 9 | 3.80379 |
| 6 | 3.30966 | 10 | 3.85924 |

We note from the above table that the $eff(\hat{\theta}_{RSS,L},\hat{\theta}_{b,L}) \geq 1$ for all $m$=3, ..., 10 and

$eff(\hat{\theta}_{RSS,L},\hat{\theta}_{b,L})$ is increasing in $m$.

**c. Log-Normal distribution with known c.v.**

Let $(j_0, \dots, j_0)$ be the best ranked set sampling scheme for $LN(\theta, \sigma^2)$ with known c.v. obtained in chapter two. The pdf $Y_{j_0:m}$ is

$$g_{j_0}(y;\theta) = \frac{C_{j_0,m}\Phi^{j_0-1}\left(\frac{logy-\theta}{\sqrt{log\,(c^2+1)}}\right)}{y\sqrt{log\,(c^2+1)}\sqrt{2\pi}}\Phi^{m-j_0}\left(\frac{\theta-logy}{\sqrt{log\,(c^2+1)}}\right)e^{-\frac{(logy-\theta)^2}{2log\,(c^2+1)}}$$

and the mean of $Y_{j_0:m}$ is

$$E(logY_{j_0:m}) = \int_0^\infty logy\frac{C_{j_0,m}\Phi^{j_0-1}\left(\frac{logy-\theta}{\sqrt{log\,(c^2+1)}}\right)}{y\sqrt{log\,(c^2+1)}\sqrt{2\pi}}e^{-\frac{(logy-\theta)^2}{2log\,(c^2+1)}}\times$$

$$\Phi^{m-j_0}\left(\frac{\theta-logy}{\sqrt{log\,(c^2+1)}}\right)dy,$$

$$= \int_{-\infty}^{\infty}(\sqrt{log\,(c^2+1)}v+\theta)\frac{C_{j_0,m}\Phi^{j_0-1}(v)}{\sqrt{2\pi}}\Phi^{m-j_0}(-v)\,e^{-\frac{v^2}{2}}\,dv,$$

$$= \int_{-\infty}^{\infty}\sqrt{log\,(c^2+1)}v\frac{C_{j_0,m}\Phi^{j_0-1}(v)}{\sqrt{2\pi}}\Phi^{m-j_0}(-v)\,e^{-\frac{v^2}{2}}\,dv +$$

$$\theta \int_{-\infty}^{\infty} \frac{C_{j_0,m}\Phi^{j_0-1}(v)}{\sqrt{2\pi}} \Phi^{m-j_0}(-v)\, e^{-\frac{v^2}{2}}\, dv,$$

$$= \theta + X_{mj_0},$$

where $X_{mj_0} = \int_{-\infty}^{\infty} \sqrt{\log\,(c^2+1)}\,v\, \frac{C_{j_0,m}\Phi^{j_0-1}(v)}{\sqrt{2\pi}} \Phi^{m-j_0}(-v)\, e^{-\frac{v^2}{2}}\, dv$. Hence,

$$E\left(\log Y_{j_0:m} - X_{mj_0}\right) = \theta.$$

So an unbiased estimator of $\theta$ based on $Y_{j_0:m}^{(1)}, \ldots, Y_{j_0:m}^{(m)}$, $m$ independent copies of $Y_{j_0:m}$, is

$$\hat{\theta}_{b,LN} = \frac{1}{m}\sum_{i=1}^{m}\left(\log Y_{j_0:m}^{(i)} - X_{mj_0}\right)$$

with variance given by

$$Var\left(\hat{\theta}_{b,C}\right) = \frac{1}{m}\, Var\left(\log Y_{j_0:m}^{(1)}\right),$$

where

$$Var\left(\log Y_{j_0:m}^{(1)}\right) = E\left(\log Y_{j_0:m}^{(1)\,2}\right) - \left(E\log Y_{j_0:m}^{(1)}\right)^2$$

$$= \int_{-\infty}^{\infty} \left(\sqrt{\log\,(c^2+1)}\,v + \theta\right)^2 \frac{C_{j_0,m}\Phi^{j_0-1}(v)}{\sqrt{2\pi}} \Phi^{m-j_0}(-v)\, e^{-\frac{v^2}{2}}\, dv -$$

$$\left(\int_{-\infty}^{\infty} \left(\sqrt{\log\,(c^2+1)}\,v + \theta\right) \frac{C_{j_0,m}\Phi^{j_0-1}(v)}{\sqrt{2\pi}} \Phi^{m-j_0}(-v)\, e^{-\frac{v^2}{2}}\, dv\right)^2$$

Under McIntyre' s RSS scheme, an estimator of $\theta$ is given by

$$\hat{\theta}_{RSS,LN} = \frac{1}{m}\sum_{j=1}^{m}\left(\log Y_{j:m} - X_{mj}\right)$$

with variance given by

$$Var(\hat{\theta}_{RSS,LN}) = \frac{1}{m^2}\sum_{j=1}^{m}\sigma_{j:m}^2,$$

where

$$\sigma_{j:m}^2 = \int_{-\infty}^{\infty}(\sqrt{\log(c^2+1)}v+\theta)^2\frac{C_{j,m}\Phi^{j-1}(v)}{\sqrt{2\pi}}\Phi^{m-j}(-v)\,e^{-\frac{v^2}{2}}dv -$$

$$\int_{-\infty}^{\infty}(\sqrt{\log(c^2+1)}v+\theta)\frac{C_{j,m}\Phi^{j-1}(v)}{\sqrt{2\pi}}\Phi^{m-j}(-v)\,e^{-\frac{v^2}{2}}dv.$$

The efficiency of $\hat{\theta}_{b,L}$ with respect to $\hat{\theta}_{RSS,L}$

$$eff(\hat{\theta}_{RSS,LN},\hat{\theta}_{b,LN}) = \frac{\sum_{j=1}^{m}\sigma_{j:m}^2}{m\,Var\left(Y_{j_0:m}^{(1)}\right)}.$$

Table 3.8 The efficiency of $\hat{\theta}_{b,L}$ w.r.t $\hat{\theta}_{RSS,L}$ for Log-Normal with known c.v.

| $m$ | $eff(\hat{\theta}_{RSS,LN},\hat{\theta}_{b,LN})$ | $m$ | $eff(\hat{\theta}_{RSS,LN},\hat{\theta}_{b,LN})$ |
|---|---|---|---|
| 3 | 1.16463 | 7 | 1.32181 |
| 4 | 1.18208 | 8 | 1.33591 |
| 5 | 1.25853 | 9 | 1.36870 |
| 6 | 1.27494 | 10 | 1.38078 |

We note from the above table that the $eff(\hat{\theta}_{RSS,LN},\hat{\theta}_{b,LN}) \geq 1$ for all $m$=3, …, 10 and $eff(\hat{\theta}_{RSS,LN},\hat{\theta}_{b,LN})$ is increasing in $m$.

### 3.4 Comparison of MLE's

In this section, we show that the MLE obtained based on a best sampling scheme is asymptotically more efficient than its counterpart under McIntyre's RSS scheme.

Let $\hat{\theta}_{\text{RSS,M}}$ be the MLE of $\theta$ obtained based on a McIntyre ranked set sample of $r$ cycles and a set size equal $m$. Under suitable regularity (see Lehmann (1999)) conditions on $f(x;\theta)$, we have the following asymptotic result for $\hat{\theta}_{\text{RSS,M}}$:

$$\sqrt{mr}\left(\hat{\theta}_{\text{RSS,M}} - \theta\right) \xrightarrow{d} N\left(0, I^{-1}(\theta)\right) \text{ as } r \to \infty.$$

where $I(\theta) = \frac{1}{m}\sum_1^m I_{Y_{j:m}}(\theta)$. This result is a direct application of Theorem 7.6.3 of Lehmann (1999). Similarly, if $\hat{\theta}_{\text{b,M}}$ is the MLE of $\theta$ obtained based on an $rm$ copies of the best sampling scheme then

$$\sqrt{mr}\left(\hat{\theta}_{\text{b,M}} - \theta\right) \xrightarrow{d} N\left(0, \quad I^{-1}{}_{Y_{j0:m}}(\theta)\right).$$

To compare the two MLE estimators of $\theta$, we use their asymptotic efficiency which is given by

$$Aeff\left(\hat{\theta}_{\text{RSS,M}}, \hat{\theta}_{\text{b,M}}\right) = \frac{I^{-1}(\theta)}{I^{-1}{}_{Y_{j0:m}}(\theta)},$$

$$= \frac{I_{Y_{j0:m}}(\theta)}{\frac{1}{m}\sum_{j=1}^m I_{Y_{j:m}}(\theta).}.$$

Provided that the last ratio is free of $\theta$ (this may occurs when $I_{Y_{j:m}}(\theta) \propto g(\theta)$ for all), it is straight forward to show that the last ratio is free of $\theta$ and is greater than or equal to one. Hence the MLE's that we have obtained based on the best sampling scheme are more efficient than their counterparts obtained from a McIntyre RSS.

Table 3.9 The values of $Aeff\left(\hat{\theta}_{RSS,M}, \hat{\theta}_{b,M}\right)$ for different distributions.

| Distribution | $m$ | $Aeff\left(\hat{\theta}_{RSS,M}, \hat{\theta}_{b,M}\right)$ | $m$ | $Aeff\left(\hat{\theta}_{RSS,M}, \hat{\theta}_{b,M}\right)$ |
|---|---|---|---|---|
| $Cauchy(0,\sigma)$ | 3 | 1.28224 | 7 | 1.22594 |
| | 4 | 1.24650 | 8 | 1.19067 |
| | 5 | 1.29224 | 9 | 1.14484 |
| | 6 | 1.25293 | 10 | 1.11450 |
| *Cauchy* with known cv. | 3 | 1.27082 | 7 | 1.6146 |
| | 4 | 1.39715 | 8 | 1.64658 |
| | 5 | 1.43376 | 9 | 1.70621 |
| | 6 | 1.53267 | 10 | 1.7663 |
| $Cauchy(\theta, 1)$ | 3 | 1.304316 | 7 | 1.625388 |
| | 4 | 1.312990 | 8 | 1.713840 |
| | 5 | 1.488869 | 9 | 1.72030 |
| | 6 | 1.497743 | 10 | |
| $Beta(\theta, 1)$ | 3 | 1.38257 | 7 | 1.46104 |
| | 4 | 1.40593 | 8 | 1.47817 |
| | 5 | 1.39962 | 9 | 1.48404 |
| | 6 | 1.42652 | 10 | 1.48243 |
| $LL(\alpha, 1)$ | 3 | 1.2 | 7 | 1.33333 |
| | 4 | 1.2 | 8 | 1.33333 |
| | 5 | 1.28571 | 9 | 1.36364 |
| | 6 | 1.28571 | 10 | 1.36364 |
| $LL(1, \beta)$ | 3 | 1.0613 | 7 | 1.3157 |
| | 4 | 1.1374 | 8 | 1.3530 |
| | 5 | 1.2082 | 9 | 1.3813 |
| | 6 | 1.2678 | 10 | 1.4020 |
| Weible$(\gamma, 1)$ | 3 | 1.32768 | 7 | 1.78363 |
| | 4 | 1.46719 | 8 | 1.85987 |
| | 5 | 1.58911 | 9 | 1.92464 |
| | 6 | 1.69403 | 10 | 1.97961 |

## 3.5 Maximum likelihood estimators

In this section, we obtain the MLE for the parameter of interest when the sample is obtained via the best sampling scheme.

### 3.5.1 Location distributions

**a. $N(\theta, 1)$**

Let $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \ldots, Y_{j_0:m}^{(r)}$ be iid copies from $Y_{j_0:m}$. The pdf of $Y_{j_0:m}^{(i)}, i = 1, 2, \ldots, r$, is

$$g_{Y^{(i)}_{j_0:m}}(y;\,\theta) = \frac{C_{j_0,m}}{\sqrt{2\pi}}\Phi^{j_0-1}(y-\theta)\quad \Phi^{m-j_0}(\theta-y)e^{-\frac{(y-\theta)^2}{2}}.$$

And the log likelihood of $Y^{(1)}_{j_0:m}, Y^{(2)}_{j_0:m}, \ldots, Y^{(r)}_{j_0:m}$ is

$$log\prod_{i=1}^{r} g_{Y^{(i)}_{j_0:m}}(y;\,\theta) = rlog\ C_{j_0,m} + (j_0-1)\sum_{i=1}^{r} log\,\Phi\left(Y^{(i)}_{j_0:m}-\theta\right) +$$

$$(m-j_0)\sum_{i=1}^{r} log\,\Phi\left(\theta - Y^{(i)}_{j_0:m}\right) - \frac{\sum_{i=1}^{r}(Y^{(i)}_{j_0:m}-\theta)^2}{2} - \frac{r}{2}log(2\pi).$$

The derivative with respect to $\theta$ is

$$\frac{\partial logg^*}{\partial\theta} = -(j_0-1)\sum_{i=1}^{r}\frac{\phi\left(Y^{(i)}_{j_0:m}-\theta\right)}{\Phi\left(Y^{(i)}_{j_0:m}-\theta\right)} + (m-j_0)\sum_{i=1}^{r}\frac{\phi\left(\theta - Y^{(i)}_{j_0:m}\right)}{\Phi\left(\theta - Y^{(i)}_{j_0:m}\right)} + \sum_{i=1}^{r}\left(Y^{(i)}_{j_0:m}-\theta\right)$$

where $g^* = \prod_{i=1}^{r} g_{Y^{(i)}_{j_0:m}}(y;\,\theta)$.

Set the last equation equals zero and solve it for $\theta$ to get the MLE for $\theta$.

Since there is no closed form solution, then a numerical method such as Newton- Raphson is needed to obtain the values of the MLE and standard error.

**b. Logistic($\theta$, 1)**

Let $Y^{(1)}_{j_0:m}, Y^{(2)}_{j_0:m}, \ldots, Y^{(r)}_{j_0:m}$ be iid copies from $Y_{j_0:m}$. The pdf of $Y^{(i)}_{j_0:m}$, $i = 1, 2, \ldots, r$, is

$$g_{Y^{(i)}_{j_0:m}}(y;\,\theta) = \frac{C_{j_0,m}e^{-(y-\theta)(m-j_0+1)}}{(1+e^{-(y-\theta)})^{m+1}}.$$

And the log likelihood of $Y^{(1)}_{j_0:m}, Y^{(2)}_{j_0:m}, \ldots, Y^{(r)}_{j_0:m}$ is

$$\log g^* = r\log C_{j_0,m} - \sum_{i=1}^{r}\left(Y_{j_0:m}^{(i)} - \theta\right)(m - j_0 + 1) - (m+1)\sum_{i=1}^{r}\log\left(1 + e^{-\left(Y_{j_0:m}^{(i)} - \theta\right)}\right),$$

The derivative with respect to $\theta$ is

$$\frac{\partial \log g^*}{\partial\theta} = (m - j_0 + 1)r + \sum_{i=1}^{r}\frac{(m+1)e^{-\left(Y_{j_0:m}^{(i)} - \theta\right)}}{\left(1 + e^{-\left(Y_{j_0:m}^{(i)} - \theta\right)}\right)}.$$

Setting the last equation equal zero and solving for $\theta$ yields the equation

$$\frac{-(m - j_0 + 1)r}{(m+1)} = \sum_{i=1}^{r}\frac{1}{\left(1 + e^{\left(Y_{j_0:m}^{(i)} - \theta\right)}\right)},$$

$$\frac{j_0 r}{(m+1)} - r = \sum_{i=1}^{r}\left(1 + e^{\left(Y_{j_0:m}^{(i)} - \theta\right)}\right)^{-1}.$$

The MLE is the solution of the last equation for $\theta$.

**c. Cauchy($\theta$, 1)**

Let $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \ldots, Y_{j_0:m}^{(r)}$ be iid copies from $Y_{j_0:m}$. The pdf of $Y_{j_0:m}^{(i)}, i = 1, 2, \ldots, r$, is

$$g_{Y_{j_0:m}^{(i)}}(y;\,\theta) = \frac{C_{j_0,m}\left(\frac{1}{\pi}\tan^{-1}(y - \theta) + \frac{1}{2}\right)^{j_0 - 1}}{\pi(1 + (y - \theta)^2)}\left(\frac{1}{2} - \frac{1}{\pi}\tan^{-1}(y - \theta)\right)^{m - j_0}$$

And the log likelihood of $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \ldots, Y_{j_0:m}^{(r)}$ is

$$\log g^* = r\log\frac{C_{j,m}}{\pi} + (j_0 - 1)\sum_{i=1}^{r}\log\left(\frac{1}{\pi}\tan^{-1}\left(Y_{j_0:m}^{(i)} - \theta\right) + \frac{1}{2}\right) +$$

$$(m - j_0)\sum_{i=1}^{r}\log\left(\frac{1}{2} - \frac{1}{\pi}\tan^{-1}\left(Y_{j_0:m}^{(i)} - \theta\right)\right) - \sum_{i=1}^{r}\log\left(1 + \left(Y_{j_0:m}^{(i)} - \theta\right)^2\right),$$

The first derivative with respect to $\theta$ is

$$\frac{\partial logg^*}{\partial\theta}=\sum_{i=1}^{r}\frac{-\frac{(j_0-1)}{\pi\left(1+\left(Y_{j_0:m}^{(i)}-\theta\right)^2\right)}}{\left(\frac{1}{\pi}tan^{-1}\left(Y_{j_0:m}^{(i)}-\theta\right)+\frac{1}{2}\right)}+\sum_{i=1}^{r}\frac{\frac{(m-j_0)}{\pi\left(1+\left(Y_{j_0:m}^{(i)}-\theta\right)^2\right)}}{\left(\frac{1}{2}-\frac{1}{\pi}tan^{-1}\left(Y_{j_0:m}^{(i)}-\theta\right)\right)}+$$

$$\sum_{i=1}^{r}\frac{2\left(Y_{j_0:m}^{(i)}-\theta\right)}{\left(1+\left(Y_{j_0:m}^{(i)}-\theta\right)^2\right)}.$$

Solving $\frac{\partial logg^*}{\partial\theta}=0$ w. r. to $\theta$. i. e.,

$$\sum_{i=1}^{r}\frac{-\frac{(j_0-1)}{\left(1+\left(Y_{j_0:m}^{(i)}-\theta\right)^2\right)}}{\left(tan^{-1}\left(Y_{j_0:m}^{(i)}-\theta\right)+\frac{\pi}{2}\right)}+\sum_{i=1}^{r}\frac{\frac{(m-j_0)}{\left(1+\left(Y_{j_0:m}^{(i)}-\theta\right)^2\right)}}{\left(\frac{\pi}{2}-tan^{-1}\left(Y_{j_0:m}^{(i)}-\theta\right)\right)}=\sum_{i=1}^{r}\frac{-2\left(Y_{j_0:m}^{(i)}-\theta\right)}{\left(1+\left(Y_{j_0:m}^{(i)}-\theta\right)^2\right)}$$

Yields the MLE of $\theta$.

**d.** **LN$(\theta,1)$**

Let $Y_{j_0:m}^{(1)},Y_{j_0:m}^{(2)},\dots,Y_{j_0:m}^{(r)}$ be iid copies from $Y_{j_0:m}$. The pdf of $Y_{j_0:m}^{(i)}, i=1,2,\dots,r,$ is

$$g_{Y_{j_0:m}^{(i)}}(y;\,\theta)=\frac{C_{j_0,m}}{Y_{j_0:m}^{(i)}\sqrt{2\pi}}\Phi^{j_0-1}\left(logY_{j_0:m}^{(i)}-\theta\right)\Phi^{m-j_0}\left(\theta-logY_{j_0:m}^{(i)}\right)e^{-\frac{1}{2}(logY_{j_0:m}^{(i)}-\theta)^2},$$

The log likelihood of $Y_{j_0:m}^{(1)},Y_{j_0:m}^{(2)},\dots,Y_{j_0:m}^{(r)}$ is

$$logg^*=\ rlog\left(\frac{C_{j_0,m}}{\sqrt{2\pi}}\right)+(j_0-1)\sum_{i=1}^{r}\log\Phi\left(logY_{j_0:m}^{(i)}-\theta\right)-\sum_{i=1}^{r}log\,Y_{j_0:m}^{(i)}+$$

$$(m-j_0)\sum_{i=1}^{r}\log\Phi\left(\theta-logY_{j_0:m}^{(i)}\right)-\sum_{i=1}^{r}\frac{(logY_{j_0:m}^{(i)}-\theta)^2}{2}.$$

$$\frac{\partial logg^*}{\partial\theta}=\sum_{i=1}^{r}\frac{-(j_0-1)\phi\left(logY_{j_0:m}^{(i)}-\theta\right)}{\Phi\left(logY_{j_0:m}^{(i)}-\theta\right)}+\sum_{i=1}^{r}(logY_{j_0:m}^{(i)}-\theta)+$$

$$\sum_{i=1}^{r}\frac{(m-j_0)\phi\left(\theta-logY_{j_0:m}^{(i)}\right)}{\Phi\left(\theta-logY_{j_0:m}^{(i)}\right)}$$

Setting the last equation equal zero and solving for $\theta$ yields the equation

$$\sum_{i=1}^{r}\frac{(j_0-1)\phi\left(logY_{j_0:m}^{(i)}-\theta\right)}{\Phi\left(logY_{j_0:m}^{(i)}-\theta\right)}-\sum_{i=1}^{r}\left(logY_{j_0:m}^{(i)}-\theta\right)=\sum_{i=1}^{r}\frac{(m-j_0)\phi\left(\theta-logY_{j_0:m}^{(i)}\right)}{\Phi\left(\theta-logY_{j_0:m}^{(i)}\right)}$$

The MLE is the solution of the last equation for $\theta$.

### 3.5.2 Scale distributions

**a. Exp ($\theta$).**

Let $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \dots, Y_{j_0:m}^{(r)}$ be iid copies from $Y_{j_0:m}$. The pdf of $Y_{j_0:m}^{(i)}, i=1,2,\dots,r$, is

$$g_{Y_{j_0:m}}(y)=C_{j_0,m}\left(1-e^{-\frac{y}{\theta}}\right)^{j_0-1}e^{-\frac{(m+1-j_0)y}{\theta}}\frac{1}{\theta},$$

and the log likelihood function is

$$\log g^*=r\log C_{j_0,m}-r\log\theta+(j_0-1)\sum_{i=1}^{r}log\left(1-e^{-\frac{Y_{j_0:m}^{(i)}}{\theta}}\right)-K_{j_0:m}\frac{\sum_{i=1}^{m}Y_{j_0:m}^{(i)}}{\theta},$$

where $K_{j_0:m}=m-j_0+1$.

Taking the first derivation with respect to $\theta$

$$\frac{\partial \log g^*}{\partial \theta} = -\frac{r}{\theta} - \frac{(j_0-1)}{\theta^2}\sum_{i=1}^{r}\frac{Y_{j_0:m}^{(i)}\, e^{-\frac{Y_{j_0:m}^{(i)}}{\theta}}}{1-e^{-\frac{Y_{j_0:m}^{(i)}}{\theta}}} + K_{j_0:m}\frac{\sum_{i=1}^{r} Y_{j_0:m}^{(i)}}{\theta^2}.$$

Setting the last equation equal zero and solving for $\theta$ yields the equation

$$(m-j_0+1)\sum_{i=1}^{r} Y_{j_0:m}^{(i)} - (j_0-1)\sum_{i=1}^{r}\frac{Y_{j_0:m}^{(i)} e^{-\frac{Y_{j_0:m}^{(i)}}{\theta}}}{1-e^{-\frac{Y_{j_0:m}^{(i)}}{\theta}}} = r\theta.$$

The MLE of $\theta$ is the solution of the last equation.

Now for $j_0 = m$, we get the following equation for MLE

$$\sum_{i=1}^{r} Y_{j_0:m}^{(i)} - (m-1)\sum_{i=1}^{r}\frac{Y_{j_0:m}^{(i)}\, e^{-\frac{Y_{j_0:m}^{(i)}}{\theta}}}{1-e^{-\frac{Y_{j_0:m}^{(i)}}{\theta}}} = r\theta.$$

and for $j_0 = m-1$ we get the following equation

$$2\sum_{i=1}^{r} Y_{j_0:m}^{(i)} - (m-2)\sum_{i=1}^{r}\frac{Y_{j_0:m}^{(i)}\, e^{-\frac{Y_{j_0:m}^{(i)}}{\theta}}}{1-e^{-\frac{Y_{j_0:m}^{(i)}}{\theta}}} = r\theta.$$

**a.** $N(0,\sigma^2)$

Let $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \ldots, Y_{j_0:m}^{(r)}$ be iid copies from $Y_{j_0:m}$. The pdf of $Y_{j_0:m}^{(i)}, i = 1,2,\ldots,r$, is

$$g_{Y_{j:m}}(y;\,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} C_{j_0,m}\Phi^{j_0-1}\left(\frac{y}{\sigma}\right)\Phi^{m-j_0}\left(-\frac{y}{\sigma}\right) e^{-\frac{y^2}{2\sigma^2}}$$

And the log likelihood of $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \ldots, Y_{j_0:m}^{(r)}$ is

$$\log g^* = r\log C_{j_0,m} + (j_0-1)\sum_{i=1}^{r} \log\Phi\left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right) + (m-j_0)\sum_{i=1}^{r} \log\Phi\left(-\frac{Y_{j_0:m}^{(i)}}{\sigma}\right) - \frac{\sum_{i=1}^{r}\left(Y_{j_0:m}^{(i)}\right)^2}{2\sigma^2} - r\log\sigma - \frac{r}{2}\log 2\pi,$$

The derivative w. r. to $\sigma^2$ is

$$\frac{\partial \log g^*}{\partial\sigma^2} = -(j_0-1)\sum_{i=1}^{r}\frac{\phi\left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right)\frac{Y_{j_0:m}^{(i)}}{2\sigma^3}}{\Phi\left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right)} + (m-j_0)\sum_{i=1}^{r}\frac{\phi\left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right)\frac{Y_{j_0:m}^{(i)}}{2\sigma^3}}{\Phi\left(-\frac{y_i}{\sigma}\right)} - \frac{r}{2\sigma^2} + \frac{\sum_{i=1}^{r}\left(Y_{j_0:m}^{(i)}\right)^2}{2\sigma^4}.$$

If we set $\frac{\partial \log g^*}{\partial\sigma^2}$ to zero, we get the equation

$$-\frac{(j_0-1)}{2}\sum_{i=1}^{r}\frac{e^{-\frac{\left(Y_{j_0:m}^{(i)}\right)^2}{\sigma^2}}Y_{j_0:m}^{(i)}}{\Phi\left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right)} + \frac{(m-j_0)}{2}\sum_{i=1}^{r}\frac{e^{-\frac{\left(Y_{j_0:m}^{(i)}\right)^2}{\sigma^2}}Y_{j_0:m}^{(i)}}{\Phi\left(-\frac{Y_{j_0:m}^{(i)}}{\sigma}\right)} + \frac{\sum_{i=1}^{r}\left(Y_{j_0:m}^{(i)}\right)^2}{2\sigma} = \frac{r\sigma}{2},$$

Where its solution to $\sigma$ is the MLE of $\sigma$.

**c. Logistic (0, $\sigma$)**

Let $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \dots, Y_{j_0:m}^{(r)}$ be iid copies from $Y_{j_0:m}$. The pdf of $Y_{j_0:m}^{(i)}, i = 1,2,\dots,r$, is

$$g_{Y_{j_0:m}^{(i)}}(y;\ \sigma) = \frac{C_{j_0,m}e^{-\frac{y(m-j_0+1)}{\sigma}}}{\sigma\left(1+e^{-\frac{y}{\sigma}}\right)^{m+1}}.$$

Taking the first derivative to the log- likelihood function with respect to $\sigma$, then we get

$$\frac{\partial \log g^*}{\partial \sigma} = -\frac{r}{\sigma} + \frac{(m - j_0 + 1)}{\sigma^2} \sum_{i=1}^{r} Y_{j_0:m}^{(i)} - \sum_{i=1}^{r} \frac{(m+1) Y_{j_0:m}^{(i)} e^{-\frac{Y_{j_0:m}^{(i)}}{\sigma}}}{\sigma^2 \left(1 + e^{-\frac{Y_{j_0:m}^{(i)}}{\sigma}}\right)}.$$

Setting the last equation equal zero and solving for $\sigma$ yields the equation

$$(m - j_0 + 1) \sum_{i=1}^{r} Y_{j_0:m}^{(i)} - (m+1) \sum_{i=1}^{r} \frac{Y_{j_0:m}^{(i)}}{\left(1 + e^{-\frac{Y_{j_0:m}^{(i)}}{\sigma}}\right)} = \sigma r.$$

The MLE is the solution of the last equation for $\sigma$.

**e. Cauchy (0, $\sigma$)**

Let $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \dots, Y_{j_0:m}^{(r)}$ be iid copies from $Y_{j_0:m}$. The pdf of $Y_{j_0:m}^{(i)}, i = 1, 2, \dots, r$, is

$$g_{Y_{j_0:m}^{(i)}}(y; \sigma) = \frac{C_{j_0,m} \left(\frac{1}{\pi} tan^{-1}\left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right) + \frac{1}{2}\right)^{j_0 - 1}}{\pi\sigma \left(1 + \left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right)^2\right)} \left(\frac{1}{2} - \frac{1}{\pi} tan^{-1}\left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right)\right)^{m - j_0}.$$

And the log likelihood of $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \dots, Y_{j_0:m}^{(r)}$ is

$$logg^* = rlog \frac{C_{j_0,m}}{\pi} - rlog\, \sigma + (j_0 - 1) \sum_{i=1}^{r} log \left(\frac{1}{\pi} tan^{-1}\left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right) + \frac{1}{2}\right) +$$

$$(m - j_0) \sum_{i=1}^{r} log \left(\frac{1}{2} - \frac{1}{\pi} tan^{-1}\left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right)\right) - \sum_{i=1}^{r} log \left(1 + \left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right)^2\right).$$

The derivative with respect to $\sigma$ is

$$\frac{\partial \log g^*}{\partial \sigma} = -\frac{r}{\sigma} - \sum_{i=1}^{r} \frac{\dfrac{Y_{j_0:m}^{(i)}(j_0-1)}{\pi\sigma^2\left(1+\left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right)^2\right)}}{\left(\frac{1}{\pi}\tan^{-1}\left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right)+\frac{1}{2}\right)} + \sum_{i=1}^{r} \frac{\dfrac{Y_{j_0:m}^{(i)}(m-j_0)}{\pi\sigma^2\left(1+\left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right)^2\right)}}{\left(\frac{1}{2}-\frac{1}{\pi}\tan^{-1}\left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right)\right)} +$$

$$\sum_{i=1}^{r} \frac{2\left(Y_{j_0:m}^{(i)}\right)^2}{\sigma^3\left(1+\left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right)^2\right)}.$$

Setting the last equation equal zero and solving for $\sigma$ yields the equation

$$r\sigma + \sum_{i=1}^{r} \frac{\dfrac{Y_{j_0:m}^{(i)}(j_0-1)}{\pi\left(1+\left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right)^2\right)}}{\left(\frac{1}{\pi}\tan^{-1}\left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right)+\frac{1}{2}\right)} - \sum_{i=1}^{r} \frac{\dfrac{Y_{j_0:m}^{(i)}(m-j_0)}{\pi\left(1+\left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right)^2\right)}}{\left(\frac{1}{2}-\frac{1}{\pi}\tan^{-1}\left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right)\right)} = \sum_{i=1}^{r} \frac{2\left(Y_{j_0:m}^{(i)}\right)^2}{\sigma\left(1+\left(\frac{Y_{j_0:m}^{(i)}}{\sigma}\right)^2\right)}$$

The MLE is the solution of the last equation for $\sigma$.

**f. LN(0, $\sigma^2$ )**

Let $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \dots, Y_{j_0:m}^{(r)}$ be iid copies from $Y_{j_0:m}$. The pdf of $Y_{j_0:m}^{(i)}, i = 1, 2, \dots, r$, is

$$g_{Y_{j_0:m}^{(i)}}(y, \sigma) = \frac{C_{j_0,m}}{y\sigma\sqrt{2\pi}} \Phi^{j_0-1}\left(\frac{\log y}{\sigma}\right) \Phi^{m-j_0}\left(-\frac{\log y}{\sigma}\right) e^{-\frac{1}{2\sigma^2}(\log y)^2}$$

The derivative for log likelihood function with respect to $\sigma$ is

$$\frac{\partial logg^*}{\partial \sigma} = -\frac{r}{\sigma} - \sum_{i=1}^{r} \frac{(j_0 - 1)logY_{j_0:m}^{(i)}\phi\left(\frac{logY_{j_0:m}^{(i)}}{\sigma}\right)}{\sigma^2\Phi\left(\frac{logY_{j_0:m}^{(i)}}{\sigma}\right)} + \sum_{i=1}^{r} \frac{(logY_{j_0:m}^{(i)})^2}{\sigma^3} +$$

$$\sum_{i=1}^{r} \frac{(m - j_0)logY_{j_0:m}^{(i)}\phi\left(-\frac{logY_{j_0:m}^{(i)}}{\sigma}\right)}{\sigma^2\Phi\left(-\frac{logY_{j_0:m}^{(i)}}{\sigma}\right)}.$$

Setting the last equation equal zero and solving for $\sigma$ yields the equation

$$-r\sigma - \sum_{i=1}^{r} \frac{(j_0 - 1)logY_{j_0:m}^{(i)}\phi\left(\frac{logY_{j_0:m}^{(i)}}{\sigma}\right)}{\Phi\left(\frac{logY_{j_0:m}^{(i)}}{\sigma}\right)} + \sum_{i=1}^{r} \frac{(logY_{j_0:m}^{(i)})^2}{\sigma} =$$

$$\sum_{i=1}^{r} \frac{-(m - j_0)logY_{j_0:m}^{(i)}\phi\left(-\frac{logY_{j_0:m}^{(i)}}{\sigma}\right)}{\Phi\left(-\frac{logY_{j_0:m}^{(i)}}{\sigma}\right)},$$

The MLE is the solution of the last equation for $\sigma$.

### 3.5.2 Location- scale distributions with known coefficient of variation

**a.** $N(\theta, \theta^2), \theta > 0$

Let $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \ldots, Y_{j_0:m}^{(r)}$ be iid copies from $Y_{j_0:m}$. The pdf of $Y_{j_0:m}^{(i)}, i = 1, 2, \ldots, r$, is

$$g_{Y_{j_0:m}^{(i)}}(y;\theta) = \frac{C_{j_0,m}}{\sqrt{2\pi}} \Phi^{j_0-1}\left(\frac{y-\theta}{\theta}\right) \Phi^{m-j_0}\left(\frac{\theta - y}{\theta}\right) \phi\left(\frac{y-\theta}{\theta}\right) \frac{1}{\theta}$$

And the log likelihood of $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \ldots, Y_{j_0:m}^{(r)}$ is

$$\log g^* = r\log C_{j_0,m} + (j_0-1)\sum_{i=1}^{r} \log\Phi\left(\frac{Y_{j_0:m}^{(i)}-\theta}{\theta}\right) + (m-j_0)\sum_{i=1}^{r} \log\Phi\left(\frac{\theta - Y_{j_0:m}^{(i)}}{\theta}\right) -$$

$$\frac{\sum_{i=1}^{r}(Y_{j_0:m}^{(i)}-\theta)^2}{2\theta^2} - r\log\theta - \frac{r}{2}\log(2\pi),$$

$$\frac{\partial \log g^*}{\partial\theta} = -(j_0-1)\sum_{i=1}^{r}\frac{\phi\left(\frac{Y_{j_0:m}^{(i)}-\theta}{\theta}\right)\frac{Y_{j_0:m}^{(i)}}{\theta^2}}{\Phi\left(\frac{Y_{j_0:m}^{(i)}-\theta}{\theta}\right)} + (m-j_0)\sum_{i=1}^{r}\frac{\phi\left(\frac{\theta-Y_{j_0:m}^{(i)}}{\theta}\right)\frac{Y_{j_0:m}^{(i)}}{\theta^2}}{\Phi\left(\frac{\theta-Y_{j_0:m}^{(i)}}{\theta}\right)} - \frac{r}{\theta} +$$

$$\frac{2(2\theta^2)\sum_{i=1}^{r}\left(Y_{j_0:m}^{(i)}-\theta\right) + 4\theta\sum_{i=1}^{r}(Y_{j_0:m}^{(i)}-\theta)^2}{4\theta^4}.$$

Setting the left hand side of the last equation to zero yields the equation

$$-(j_0-1)\sum_{i=1}^{r}\frac{\phi\left(\frac{Y_{j_0:m}^{(i)}-\theta}{\theta}\right)Y_{j_0:m}^{(i)}}{\Phi\left(Y_{j_0:m}^{(i)}-\theta\right)} + (m-j_0)\sum_{i=1}^{r}\frac{\phi\left(\frac{\theta-Y_{j_0:m}^{(i)}}{\theta}\right)Y_{j_0:m}^{(i)}}{\Phi\left(\theta-Y_{j_0:m}^{(i)}\right)} +$$

$$\frac{\sum_{i=1}^{r}(Y_{j_0:m}^{(i)}-\theta)^2}{\theta} = 2r\theta - r\bar{y}$$

The MLE is the solution of the last equation for $\theta$.

**b. Logistic($\theta,\sigma$) with known c.v.**

Let $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \ldots, Y_{j_0:m}^{(r)}$ be iid copies from $Y_{j_0:m}$. The pdf of $Y_{j_0:m}^{(i)}, i = 1,2,\ldots,r$, is

$$g_{Y_{j_0:m}^{(i)}}(y;\,\theta) = \frac{\pi C_{j_0,m} e^{\frac{\pi(y-\theta)(m-j_0+1)}{c\sqrt{3}\theta}}}{c\sqrt{3}\theta\left(1+e^{-\frac{(y-\theta)\pi}{c\sqrt{3}\theta}}\right)^{m+1}}.$$

And the log likelihood of $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \ldots, Y_{j_0:m}^{(r)}$ is

$$\log g^* = r\log\left(\frac{\pi C_{j_0,m}}{c\sqrt{3}}\right) - r\log\theta - \sum_{i=1}^{r}\frac{\pi\left(Y_{j_0:m}^{(i)} - \theta\right)(m - j_0 + 1)}{c\sqrt{3}\theta} -$$

$$(m+1)\sum_{i=1}^{r}\log\left(1 + e^{\frac{(Y_{j_0:m}^{(i)} - \theta)\pi}{c\theta\sqrt{3}}}\right),$$

Taking the first derivative with respect to $\theta$

$$\frac{\partial \log g^*}{\partial\theta} = -\frac{r}{\theta} + \frac{(m - j_0 + 1)\pi}{c\sqrt{3}}\sum_{i=1}^{r}\frac{Y_{j_0:m}^{(i)}}{\theta^2} - \sum_{i=1}^{r}\frac{(m+1)\pi Y_{j_0:m}^{(i)} e^{\frac{(Y_{j_0:m}^{(i)} - \theta)\pi}{c\theta\sqrt{3}}}}{c\sqrt{3}\theta^2\left(1 + e^{\frac{(Y_{j_0:m}^{(i)} - \theta)\pi}{c\theta\sqrt{3}}}\right)}.$$

Setting the last equation equal zero and solving for $\theta$ yields the equation

$$(m - j_0 + 1)\pi\sum_{i=1}^{r} Y_{j_0:m}^{(i)} - (m+1)\pi\sum_{i=1}^{r}\frac{Y_{j_0:m}^{(i)}}{\left(e^{\frac{(Y_{j_0:m}^{(i)} - \theta)\pi}{c\theta\sqrt{3}}} + 1\right)} = c\sqrt{3}\theta r$$

The MLE is the solution of the last equation for $\theta$.

**c. Cauchy distribution with known c.v.**

Let $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \ldots, Y_{j_0:m}^{(r)}$ be iid copies from $Y_{j_0:m}$. The pdf of $Y_{j_0:m}^{(i)}, i = 1, 2, \ldots, r$, is

$$g_{Y_{j_0:m}}(y;\ \theta) = \frac{C_{j,m}\left(\frac{1}{2} - \frac{1}{\pi}\tan^{-1}\left(\frac{y - \theta}{c\theta}\right)\right)^{m - j_0}}{\pi c\theta\left(1 + \left(\frac{y - \theta}{c\theta}\right)^2\right)}\left(\frac{1}{\pi}\tan^{-1}\left(\frac{y - \theta}{c\theta}\right) + \frac{1}{2}\right)^{j_0 - 1}.$$

And the log likelihood of $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \ldots, Y_{j_0:m}^{(r)}$ is

$$\log g^* = r\log\frac{C_{j_0,m}}{\pi c} - r\log\theta + (j_0-1)\sum_{i=1}^{r}\log\left(\frac{1}{\pi}\tan^{-1}\left(\frac{Y_{j_0:m}^{(i)}-\theta}{c\theta}\right)+\frac{1}{2}\right)+$$

$$(m-j_0)\sum_{i=1}^{r}\log\left(\frac{1}{2}-\frac{1}{\pi}\tan^{-1}\left(\frac{Y_{j_0:m}^{(i)}-\theta}{c\theta}\right)\right)-\sum_{i=1}^{r}\log\left(1+\left(\frac{Y_{j_0:m}^{(i)}-\theta}{c\theta}\right)^2\right).$$

Taking the first derivative with respect to $\theta$

$$\frac{\partial \log g^*}{\partial\theta} = -\frac{r}{\theta} - \sum_{i=1}^{r}\frac{\dfrac{(j_0-1)Y_{j_0:m}^{(i)}}{\pi c\theta^2\left(1+\left(\frac{Y_{j_0:m}^{(i)}-\theta}{c\theta}\right)^2\right)}}{\left(\frac{1}{\pi}\tan^{-1}\left(\frac{Y_{j_0:m}^{(i)}-\theta}{c\theta}\right)+\frac{1}{2}\right)} + \sum_{i=1}^{r}\frac{\dfrac{(m-j_0)Y_{j_0:m}^{(i)}}{\pi c\theta^2\left(1+\left(\frac{Y_{j_0:m}^{(i)}-\theta}{c\theta}\right)^2\right)}}{\left(\frac{1}{2}-\frac{1}{\pi}\tan^{-1}\left(\frac{Y_{j_0:m}^{(i)}-\theta}{c\theta}\right)\right)}+$$

$$\sum_{i=1}^{r}\frac{2\left(\frac{Y_{j_0:m}^{(i)}-\theta}{c\theta}\right)\frac{Y_{j_0:m}^{(i)}}{\theta^2 c}}{\left(1+\left(\frac{Y_{j_0:m}^{(i)}-\theta}{c\theta}\right)^2\right)}.$$

Setting the last equation equal zero and solving for $\theta$ yields the equation

$$r\theta + \sum_{i=1}^{r}\frac{\dfrac{(j_0-1)Y_{j_0:m}^{(i)}}{\pi c\left(1+\left(\frac{Y_{j_0:m}^{(i)}-\theta}{c\theta}\right)^2\right)}}{\left(\frac{1}{\pi}\tan^{-1}\left(\frac{Y_{j_0:m}^{(i)}-\theta}{c\theta}\right)+\frac{1}{2}\right)} - \sum_{i=1}^{r}\frac{\dfrac{(m-j_0)Y_{j_0:m}^{(i)}}{\pi c\left(1+\left(\frac{Y_{j_0:m}^{(i)}-\theta}{c\theta}\right)^2\right)}}{\left(\frac{1}{2}-\frac{1}{\pi}\tan^{-1}\left(\frac{Y_{j_0:m}^{(i)}-\theta}{c\theta}\right)\right)} =$$

$$\sum_{i=1}^{r} \frac{2\left(Y_{j_0:m}^{(i)}-\theta\right)\frac{Y_{j_0:m}^{(i)}}{c}}{\left(1+\left(\frac{Y_{j_0:m}^{(i)}-\theta}{c\theta}\right)^2\right)}.$$

The MLE is the solution of the last equation for $\theta$.

**d. Log- normal with known c.v.**

Let $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \dots, Y_{j_0:m}^{(r)}$ be iid copies from $Y_{j_0:m}$. The pdf of $Y_{j_0:m}^{(i)}, i = 1,2,\dots,r$, is

$$g_{Y_{j_0:m}^{(i)}}(y;\theta) = \frac{C_{j,m}\Phi^{j_0-1}\left(\frac{\log y-\theta}{\sqrt{\log(c^2+1)}}\right)}{y\sqrt{\log(c^2+1)}\sqrt{2\pi}}\Phi^{m-j_0}\left(\frac{\theta-\log y}{\sqrt{\log(c^2+1)}}\right)e^{-\frac{(\log y-\theta)^2}{2\log(c^2+1)}}$$

And the log likelihood of $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \dots, Y_{j_0:m}^{(r)}$ is

$$\log g^* = r\log\left(\frac{C_{j_0,m}/\sqrt{2\pi}}{\sqrt{\log(c^2+1)}}\right) + (j_0-1)\sum_{i=1}^{r}\log\Phi\left(\frac{\log Y_{j_0:m}^{(i)}-\theta}{\sqrt{\log(c^2+1)}}\right) +$$

$$(m-j_0)\sum_{i=1}^{r}\log\Phi\left(\frac{\theta-\log Y_{j_0:m}^{(i)}}{\sqrt{\log(c^2+1)}}\right) - \sum_{i=1}^{r}\frac{(\log Y_{j_0:m}^{(i)}-\theta)^2}{2\log(c^2+1)} - \sum_{i=1}^{r}\log Y_{j_0:m}^{(i)}.$$

Taking the first derivative with respect to $\theta$

$$\frac{\partial \log g^*}{\partial\theta} = -\frac{(j_0-1)}{\sqrt{\log(c^2+1)}}\sum_{i=1}^{r}\frac{\phi\left(\frac{\log Y_{j_0:m}^{(i)}-\theta}{\sqrt{\log(c^2+1)}}\right)}{\Phi\left(\frac{\log Y_{j_0:m}^{(i)}-\theta}{\sqrt{\log(c^2+1)}}\right)} + \sum_{i=1}^{r}\frac{(\log Y_{j_0:m}^{(i)}-\theta)}{\log(c^2+1)} +$$

$$\frac{(m-j_0)}{\sqrt{\log(c^2+1)}}\sum_{i=1}^{r}\frac{\phi\left(\frac{\theta-\log Y_{j_0:m}^{(i)}}{\sqrt{\log(c^2+1)}}\right)}{\Phi\left(\frac{\theta-\log Y_{j_0:m}^{(i)}}{\sqrt{\log(c^2+1)}}\right)}.$$

Setting the last equation equal zero and solving for $\theta$ yields the equation

$$(j_0-1)\sum_{i=1}^{r}\frac{\phi\left(\frac{logY_{j_0:m}^{(i)}-\theta}{\sqrt{\log(c^2+1)}}\right)}{\Phi\left(\frac{logY_{j_0:m}^{(i)}-\theta}{\sqrt{\log(c^2+1)}}\right)}-\sum_{i=1}^{r}\frac{(logY_{j_0:m}^{(i)}-\theta)}{\sqrt{log\,(c^2+1)}}=(m-j_0)\sum_{i=1}^{r}\frac{\phi\left(\frac{\theta-logY_{j_0:m}^{(i)}}{\sqrt{log\,(c^2+1)}}\right)}{\Phi\left(\frac{\theta-logY_{j_0:m}^{(i)}}{\sqrt{log\,(c^2+1)}}\right)}$$

The MLE is the solution of the last equation for $\theta$.

### 3.5.4 Other distributions

**a. Beta(θ, 1)**

Let $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \ldots, Y_{j_0:m}^{(r)}$ be iid copies from $Y_{j_0:m}$. The pdf of $Y_{j_0:m}^{(i)}, i=1,2,\ldots,r$, is

$$g_{Y_{j_0:m}^{(i)}}(y;\,\theta)=C_{j_0,m}\,\theta y^{\theta j_0-1}\left(1-y^{\theta}\right)^{m-j_0}.$$

And the log likelihood of $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \ldots, Y_{j_0:m}^{(r)}$ is

$$log\,g^*=rlog\,C_{j_0,m}+rlog\,\theta+(\theta j_0-1)\sum_{i=1}^{r}log\,Y_{j_0:m}^{(i)}+$$

$$(m-j_0)\sum_{i=1}^{r}log\left(1-\left(Y_{j_0:m}^{(i)}\right)^{\theta}\right),$$

Taking the first derivative with respect to $\theta$

$$\frac{\partial log\,g^*}{\partial\theta}=\frac{r}{\theta}+j_0\sum_{i=1}^{r}log\,Y_{j_0:m}^{(i)}-(m-j_0)\sum_{i=1}^{r}\frac{\left(Y_{j_0:m}^{(i)}\right)^{\theta}log\,Y_{j_0:m}^{(i)}}{\left(1-\left(Y_{j_0:m}^{(i)}\right)^{\theta}\right)},$$

Setting the last equation equal zero and solving for $\theta$ yields the equation

$$\frac{r}{\theta}-(m-j_0)\sum_{i=1}^{r}\frac{\left(Y_{j_0:m}^{(i)}\right)^{\theta}log\,Y_{j_0:m}^{(i)}}{\left(1-\left(Y_{j_0:m}^{(i)}\right)^{\theta}\right)}=-j_0\sum_{i=1}^{r}log\,Y_{j_0:m}^{(i)}.$$

The MLE is the solution of the last equation for $\theta$.

**b. Beta$(1,\theta)$**

Let $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \dots, Y_{j_0:m}^{(r)}$ be iid copies from $Y_{j_0:m}$. The pdf of $Y_{j_0:m}^{(i)}, i = 1, 2, \dots, r$, is

$$g_{Y_{j_0:m}^{(i)}}(y;\, \theta) = C_{j,m}\theta(1-y)^{\theta(m+1-j_0)-1}\left[1-(1-y)^{\theta}\right]^{j_0-1}.$$

And the log likelihood of $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \dots, Y_{j_0:m}^{(r)}$ is

$$\log g^* = r\log\, C_{j_0,m} + r\log\theta + (\theta(m+1-j_0)-1)\sum_{i=1}^{r}\log(1-Y_{j_0:m}^{(i)}) +$$

$$(j_0-1)\sum_{i=1}^{r}\log\left(1-\left(1-Y_{j_0:m}^{(i)}\right)^{\theta}\right).$$

Taking the first derivative with respect to $\theta$

$$\frac{\partial \log g^*}{\partial\theta} = \frac{r}{\theta} + (m+1-j_0)\sum_{i=1}^{r}\log(1-Y_{j_0:m}^{(i)}) - (j_0-1)\sum_{i=1}^{r}\frac{\log\left(1-Y_{j_0:m}^{(i)}\right)}{-1+\left(1-Y_{j_0:m}^{(i)}\right)^{-\theta}}.$$

Setting the last equation equal zero and solving for $\theta$ yields the equation

$$\frac{r}{\theta} - (j_0-1)\sum_{i=1}^{r}\frac{\log\left(1-Y_{j_0:m}^{(i)}\right)}{\left(-1+\left(1-Y_{j_0:m}^{(i)}\right)^{-\theta}\right)} = -(m+1-j_0)\sum_{i=1}^{r}\log\left(1-Y_{j_0:m}^{(i)}\right).$$

The MLE is the solution of the last equation for $\theta$.

**c. Power function distribution**

Let $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \dots, Y_{j_0:m}^{(r)}$ be iid copies from $Y_{j_0:m}$. The pdf of $Y_{j_0:m}^{(i)}, i = 1, 2, \dots, r$, is

$$g_{Y_{j_0:m}^{(i)}}(y;\,\theta) = C_{j_0,m}\frac{\theta}{1-\theta}y^{\frac{\theta(j_0+1)-1}{1-\theta}}\left(1-y^{\frac{\theta}{1-\theta}}\right)^{m-j_0}.$$

And the log likelihood of $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \dots, Y_{j_0:m}^{(r)}$ is

$$\log g^* = r\log C_{j_0,m} + r\log\theta - r\log(1-\theta) + \frac{\theta(j_0+1)-1}{1-\theta}\sum_{i=1}^{r}\log Y_{j_0:m}^{(i)} +$$

$$(m-j_0)\sum_{i=1}^{r}\log\left(1-\left(Y_{j_0:m}^{(i)}\right)^{\frac{\theta}{1-\theta}}\right).$$

Taking the first derivative with respect to $\theta$

$$\frac{\partial \log g^*}{\partial\theta} = \frac{r}{\theta} + \frac{r}{(1-\theta)} + \frac{j_0\sum_{i=1}^{r}\log Y_{j_0:m}^{(i)}}{(1-\theta)^2} - \sum_{i=1}^{r}\frac{(m-j_0)\log Y_{j_0:m}^{(i)}}{(1-\theta)^2\left(\left(Y_{j_0:m}^{(i)}\right)^{-\frac{\theta}{1-\theta}}-1\right)}.$$

Setting the last equation equal zero and solving for $\theta$ yields the equation

$$j_0\sum_{i=1}^{r}\log Y_{j_0:m}^{(i)} - (m-j_0)\sum_{i=1}^{r}\frac{\log Y_{j_0:m}^{(i)}}{\left(\left(Y_{j_0:m}^{(i)}\right)^{-\frac{\theta}{1-\theta}}-1\right)} = -\frac{r(1-\theta)}{\theta}$$

The MLE is the solution of the last equation for $\theta$.

**d. Pareto distribution**

Let $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \dots, Y_{j_0:m}^{(r)}$ be iid copies from $Y_{j_0:m}$. The pdf of $Y_{j_0:m}^{(i)}$, $i = 1,2,\dots,r$, is

$$g_{Y_{j_0:m}^{(i)}}(y;\,\beta) = \frac{C_{j_0,m}\beta\alpha^{\beta}}{y^{\beta+1}}\left(1-\left(\frac{\alpha}{y}\right)^{\beta}\right)^{j_0-1}\left(\frac{\alpha}{y}\right)^{\beta(m-j_0)}.$$

And the log likelihood of $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \dots, Y_{j_0:m}^{(r)}$ is

$$\log g^* = r\log C_{j_0,m} + r\log\beta + r\beta\log\alpha - (\beta+1)\sum_{i=1}^{r}\log Y_{j_0:m}^{(i)} +$$

$$(j_0-1)\sum_{i=1}^{r}\log\left(1-\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}\right) + \beta(m-j_0)\sum_{i=1}^{r}\log\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right),$$

Since $\alpha$ is known, we take the first derivative with respect to $\beta$

$$\frac{\partial \log g^*}{\partial\beta} = \frac{r}{\beta} + r\log\alpha - \sum_{i=1}^{r}\log Y_{j_0:m}^{(i)} - \sum_{i=1}^{r}\frac{(j_0-1)\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}\log\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)}{\left(1-\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}\right)} +$$

$$(m-j_0)\sum_{i=1}^{r}\log\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right).$$

Setting the last equation equal zero and solving for $\beta$ yields the equation

$$\frac{r}{\beta} - (j_0-1)\sum_{i=1}^{r}\frac{\log\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)}{\left(\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{-\beta}-1\right)} = -(m-j_0)\sum_{i=1}^{r}\log\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right) - r\log\alpha + \sum_{i=1}^{r}\log Y_{j_0:m}^{(i)}.$$

The MLE is the solution of the last equation for $\beta$.

**e. Weibull distribution**

Case 1. If $\gamma$ is known

Let $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \ldots, Y_{j_0:m}^{(r)}$ be iid copies from $Y_{j_0:m}$. The pdf of $Y_{j_0:m}^{(i)}, i = 1, 2, \ldots, r,$ is

$$g_{Y^{(i)}_{j_0:m}}(y;\beta)=\frac{C_{j_0,m}\gamma}{\beta}\left(1-e^{-\left(\frac{y}{\beta}\right)^{\gamma}}\right)^{j_0-1}e^{-(m-j_0+1)\left(\frac{y}{\beta}\right)^{\gamma}}\left(\frac{y}{\beta}\right)^{\gamma-1}$$

The likelihood function is

$$logg^{*}= rlog\gamma + rlog\ C_{j,m} - r\gamma log\,\beta + c_1\sum_{i=1}^{r} log\left(1-e^{-\left(\frac{Y^{(i)}_{j_0:m}}{\beta}\right)^{\gamma}}\right) -$$

$$\sum_{i=1}^{r} c_2\left(\frac{Y^{(i)}_{j_0:m}}{\beta}\right)^{\gamma} + (\gamma-1)\sum_{i=1}^{r} log\,Y^{(i)}_{j_0:m}.$$

where $c_1 = j_0 - 1$ and $c_2 = m - j_0 + 1$.

Taking the first derivative with respect to $\beta$

$$\frac{\partial logg^{*}}{\partial\beta} = -\frac{r\gamma}{\beta} - c_1\sum_{i=1}^{r}\frac{\gamma\left(Y^{(i)}_{j_0:m}\right)^{\gamma}\beta^{-(\gamma+1)}e^{-\left(\frac{Y^{(i)}_{j_0:m}}{\beta}\right)^{\gamma}}}{\left(1-e^{-\left(\frac{Y^{(i)}_{j_0:m}}{\beta}\right)^{\gamma}}\right)} + c_2\sum_{i=1}^{r}\gamma\left(Y^{(i)}_{j_0:m}\right)^{\gamma}\beta^{-(\gamma+1)}.$$

Setting the last equation equal zero and solving for $\beta$ yields the equation

$$r + c_1\beta^{-\gamma}\sum_{i=1}^{r}\frac{\left(Y^{(i)}_{j_0:m}\right)^{\gamma}}{\left(e^{\left(\frac{Y^{(i)}_{j_0:m}}{\beta}\right)^{\gamma}}-1\right)} = c_2\beta^{-\gamma}\sum_{i=1}^{r}\left(Y^{(i)}_{j_0:m}\right)^{\gamma}$$

The MLE is the solution of the above equation for $\beta$.

If $\beta$ is known, in this case we derive $logg^* = log\prod_{i=1}^{r} g_{Y_{j_0:m}}(y;\gamma)$ with respect to $\gamma$,

then we get

$$\frac{\partial logg^*}{\partial\gamma} = \frac{r}{\gamma} - rlog\beta + \sum_{i=1}^{r}\frac{c_1\left(\frac{y_i}{\beta}\right)^{\gamma} log\left(\frac{Y_{j_0:m}^{(i)}}{\beta}\right)}{e^{\left(\frac{y_i}{\beta}\right)^{\gamma}} - 1} + \sum_{i=1}^{r} log\, Y_{j_0:m}^{(i)} -$$

$$\sum_{i=1}^{r} c_2\left(\frac{Y_{j_0:m}^{(i)}}{\beta}\right)^{\gamma} log\left(\frac{Y_{j_0:m}^{(i)}}{\beta}\right),$$

Setting the last equation equal zero and solving for $\gamma$ yields the equation

$$\frac{r}{\gamma} + \sum_{i=1}^{r}\frac{c_1\left(\frac{Y_{j_0:m}^{(i)}}{\beta}\right)^{\gamma} log\left(\frac{Y_{j_0:m}^{(i)}}{\beta}\right)}{e^{\left(\frac{Y_{j_0:m}^{(i)}}{\beta}\right)^{\gamma}} - 1} - \sum_{i=1}^{r} c_2\left(\frac{Y_{j_0:m}^{(i)}}{\beta}\right)^{\gamma} log\left(\frac{Y_{j_0:m}^{(i)}}{\beta}\right) = rlog\beta - \sum_{i=1}^{r} log\, Y_{j_0:m}^{(i)}$$

The MLE is the solution of the above equation for $\gamma$.

**f. LL($\alpha,\beta$)**

$$g_{Y_{j_0:m}^{(i)}}(y;\alpha,\beta) = \frac{C_{j_0,m}\beta(y/\alpha)^{\beta-1}}{\alpha\left(1+\left(\frac{y}{\alpha}\right)^{\beta}\right)^2}\left(1-\frac{1}{1+\left(\frac{\alpha}{y}\right)^{\beta}}\right)^{m-j_0}\left(\frac{1}{1+\left(\frac{\alpha}{y}\right)^{\beta}}\right)^{j_0-1}.$$

And the log likelihood of $Y_{j_0:m}^{(1)}, Y_{j_0:m}^{(2)}, \ldots, Y_{j_0:m}^{(r)}$ is

$$logg^* = rlog\, C_{j_0,m} + (\beta-1)\sum_{i=1}^{r} log\, Y_{j_0:m}^{(i)} - (j_0-1)\sum_{i=1}^{r} log\left(1+\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}\right) -$$

$$r\beta log\alpha + rlog\,\beta + (m-j_0)\sum_{i=1}^{r} log\left(1-\frac{1}{1+\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}}\right) - 2\sum_{i=1}^{r} log\left(1+\left(\frac{Y_{j_0:m}^{(i)}}{\alpha}\right)^{\beta}\right).$$

Case 1. If $\beta$ is known

Taking the first derivative of $logg^*$ with respect to $\alpha$

$$\frac{\partial logg^*}{\partial\alpha} = -\sum_{i=1}^{r}\frac{(j_0-1)\beta\alpha^{\beta-1}}{\left(Y_{j_0:m}^{(i)}\right)^{\beta}\left(1+\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}\right)} - \frac{r\beta}{\alpha} + \sum_{i=1}^{r}\frac{2\left(Y_{j_0:m}^{(i)}\right)^{\beta}\beta\alpha^{-(1+\beta)}}{1+\left(\frac{Y_{j_0:m}^{(i)}}{\alpha}\right)^{\beta}} +$$

$$(m-j_0)\sum_{i=1}^{r}\frac{\beta\alpha^{\beta-1}\left(1+\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}\right)^{-2}}{\left(Y_{j_0:m}^{(i)}\right)^{\beta}\left(1-\frac{1}{1+\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}}\right)}.$$

Setting the last equation equal zero and solving for $\alpha$ yields the equation

$$(m-j_0)\sum_{i=1}^{r}\frac{\alpha^{\beta}\left(1+\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}\right)^{-1}}{\left(Y_{j_0:m}^{(i)}\right)^{\beta}\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}} + \sum_{i=1}^{r}\frac{2\left(Y_{j_0:m}^{(i)}\right)^{\beta}\alpha^{-\beta}}{1+\left(\frac{Y_{j_0:m}^{(i)}}{\alpha}\right)^{\beta}} =$$

$$\sum_{i=1}^{r}\frac{(j_0-1)\alpha^{\beta}}{\left(Y_{j_0:m}^{(i)}\right)^{\beta}\left(1+\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}\right)}+r$$

The MLE is the solution of the last equation for $\alpha$.

Case 2. If $\alpha$ is known

Taking the first derivative for $logg^*$ with respect to $\beta$ we get

$$\frac{\partial logg^*}{\partial\beta}=\frac{r}{\beta}-\sum_{i=1}^{r}\frac{(j_0-1)\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}log\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)}{\left(1+\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}\right)}-rlog\alpha+\sum_{i=1}^{r}Y_{j_0:m}^{(i)}+$$

$$\sum_{i=1}^{r}\frac{(m-j_0)\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}log\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)}{\left(1-\frac{1}{1+\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}}\right)\left(1+\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}\right)^{2}}-\sum_{i=1}^{r}\frac{2\left(\frac{Y_{j_0:m}^{(i)}}{\alpha}\right)^{\beta}log\left(\frac{Y_{j_0:m}^{(i)}}{\alpha}\right)}{1+\left(\frac{Y_{j_0:m}^{(i)}}{\alpha}\right)^{\beta}}.$$

Setting the last equation equal zero and solving for $\beta$ yields the equation

$$\sum_{i=1}^{r}\frac{(j_0-1)\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}log\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)}{\left(1+\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}\right)}+rlog\alpha+\sum_{i=1}^{r}\frac{2\left(\frac{Y_{j_0:m}^{(i)}}{\alpha}\right)^{\beta}log\left(\frac{Y_{j_0:m}^{(i)}}{\alpha}\right)}{1+\left(\frac{Y_{j_0:m}^{(i)}}{\alpha}\right)^{\beta}}=\frac{r}{\beta}+$$

$$\sum_{i=1}^{r} Y_{j_0:m}^{(i)} + \sum_{i=1}^{r} \frac{(m-j_0)\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta} \log\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)}{\left(1-\frac{1}{1+\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}}\right)\left(1+\left(\frac{\alpha}{Y_{j_0:m}^{(i)}}\right)^{\beta}\right)^{2}}$$

The MLE is the solution of the last equation for $\beta$.

### 3.6 Concluding remarks

Based on the numerical results, presented in this chapter, we report the following concluding remarks

1. It can be noted from the numerical results in Tables 3.1-3.8 that the estimators obtained under best ranked set sampling schemes are more efficient than their counterparts under the McIntyre's ranked set sampling scheme.

2. The efficiencies are in general monotone increasing functions of $m$.

3. For large samples size, i. e., as $r \rightarrow \infty$, Table 3.9 shows that the MLE under best sampling scheme is asymptotically more efficient than its counterpart under McIntyre's RSS scheme for all $m \geq 3$.

# Chapter four

## Conclusion and Future Work

In this thesis, we have used the Fisher information contained in order statistics to characterize the best ranked set sampling scheme for several parametric families. Also, we used these best sampling schemes to obtain unbiased and MLE estimators for the parameters of interest. These estimators are compared to their counterparts under McIntyre's scheme via their efficiencies. Numerical results showed that the estimators under best schemes are more efficient than their counter parts. Also these sampling schemes can be employed to derive more accurate inferential statements about the parameters of interest such as exact and large - sample confidence intervals. Moreover, a prediction intervals for several characteristics of future samples are also attainable. For example one may derive prediction interval for mean, order statistics of a future sample from a given parametric family.

To simplify the work, we did not give our attention to the multi-parameters case, since the maximization of Fisher information matrix is not defined in this case. However, it is still possible to attack this problem. Instead of maximizing the Fisher information, we may find the order statistic that minimizes the Cramer-Rao lower bound of the variance of any unbiased estimator. Because these works lie beyond the objectives of my thesis, I leave it to my future research.

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